The Kuramoto Model

A presentation in partial satisfaction of the requirements for the degree of MSc in Applied Mathematics

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Motivation: Phenomenon of collective synchronization – a large number of independent oscillators spontaneously synchronize to a common frequency and phase.

Examples:

- Pacemaker cells in the heart
- Metabolic cycles of yeast cells
- Crickets chirp in unison
- Synchronously flashing fireflies
The fireflies which are found in Siam are very common and are remarkable for the intensity of their light. One sees these insects fly separately from one tree to another and in all directions, but more often they are assembled by thousands on a great tree on the bank of the river. It is a magnificent spectacle to see spring out at one time from all the branches of this tree as it were, thousands of great electric sparks, because these fireflies emit not a continuous light but one interrupted by the effect of a sort of respiration. It is difficult to explain how this emission of light is simultaneous for several thousands of individuals.

Bishiop Jean Baptiste Pallegoix (1854)
The bushes literally swarm with fireflies, which flash out their intermittent light almost contemporaneously; the effect being that for an instant the exact outline of all the bushes stands prominently forward, as if lit up with electric sparks, and next moment all is jetty dark – darker from the momentary illumination that preceded. These flashes succeed one another every three or four seconds for about ten minutes, when an interval of similar duration takes place; as if to allow the insects to regain their electric or phosphoric vigor.

John Cameron (1865)
Quotations from:

Construct a Model of Coupled Oscillators

Stable limit-cycle oscillators:

Limit cycle in phase space
Two timescales:

- Oscillators relax to their limit cycles on one timescale
- Oscillators interact with one another on a second timescale

Assume first timescale $\ll$ second timescale.

This is what we mean by weak coupling.

So oscillators are always close to their limit cycles and we can describe an oscillator’s state by a phase $\theta$. 
Let the oscillator’s *natural frequency* be $\omega$.

If we have $N$ oscillators with no coupling, then

$$\dot{\theta}_i = \omega_i \quad i = 1 \ldots N$$

Kuramoto showed that for a very general system of coupled oscillators we can write

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij} (\theta_j - \theta_i)$$

where $\Gamma_{ij}$ is the *interaction function*. 
Kuramoto chose a specific interaction function

\[ \Gamma_{ij} = \frac{K}{N} \sin(\theta_j - \theta_i) \]

where \( K \) is the coupling parameter.

So the Kuramoto model is

\[ \dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \]  \hspace{1cm} (1)

This is easy to model and solve in Matlab (2 lines!).
Matlab code:

Define Kuramoto model dynamics:

\[
d\mathbf{P}/dt = @(\mathbf{P},W,c) \ W+c*\text{sum}(\sin(\text{meshgrid}(\mathbf{P})-\text{meshgrid}(\mathbf{P}')),2);
\]

Solve the system:

\[
[T \ P] = \text{ode45}(@(t,p)d\mathbf{P}/dt(p,W,c),[0 \ tmax],\mathbf{P}0);
\]
Distribution of Frequencies

Assume the natural frequencies are distributed according to some probability density $g(\omega)$, where $g$ is *symmetric* and *unimodal* about some mean frequency $\bar{\omega}$.
Redefine $\theta_i$ by $\theta_i + \bar{\omega}t$ and equation (1) becomes

$$\dot{\theta}_i + \bar{\omega} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j + \bar{\omega}t - \theta_i - \bar{\omega}t)$$

$$\dot{\theta}_i = (\omega_i - \bar{\omega}) = + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)$$

Redefine $\omega_i$ by $\omega_i - \bar{\omega}$ and we have equation (1) back again

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)$$
The distribution function $g$ becomes symmetric about zero

\[ g(\omega) = g(-\omega) \]

**Symmetric:** $g(\omega) = g(-\omega)$

**Unimodal:** nowhere increasing on the interval $[0, \infty)$
The Order Parameter

We can easily simulate a collection of $N$ oscillators using Matlab. We find that oscillators become synchronized for certain values of the coupling parameter $K$.

To get a quantitative idea of what is happening we need a measure of how synchronized the system is.
Imagine the phases as points on the unit circle

Represent the point as a complex number $e^{i\theta_i}$ and take the average over all oscillators

$$R = \frac{1}{N} \sum_{j=1}^{\infty} e^{i\theta_j}$$

This defines the complex order parameter $R$. 
Write $R$ as $re^{i\phi}$

$r$ is a measure of the *phase coherence*

$\phi$ is the *average phase*
We can rewrite equation (1) in terms of \( R \) as follows

\[
R = re^{i\phi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}
\]

Multiply by \( e^{-i\theta_i} \)

\[
r e^{i(\phi - \theta_i)} = \frac{1}{N} \sum_{j=1}^{N} e^{i(\theta_j - \theta_i)}
\]

Take the imaginary part

\[
r \sin(\phi - \theta_i) = \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)
\]
So equation (1) becomes

\[ \dot{\theta}_i = \omega_i + Kr \sin(\phi - \theta_i) \]

It looks like we have uncoupled the equations!

But we haven’t, of course.

The equations are still coupled through the values of \( r \) and \( \phi \).

The values \( r \) and \( \phi \) are “averages” of the state of the system.
So we see that an oscillator interacts with the others via an “average” of the current state of the system.

The Kuramoto model is a *mean field* model.

We see that:

- The phase $\theta_i$ of an oscillator is pulled towards the mean phase $\phi$ (steady state: $\phi = \theta_i$)

- The strength of the coupling is proportional to the coherence $r$ (positive feedback)
Time Evolution of the System

We can solve the system using Matlab.

What we find is

- For weak coupling the system evolves into a disordered (incoherent) state.

- For strong coupling the system becomes orderly but not all oscillators synchronize.
Weak Coupling

For small values of $K$, and starting in any state, the system evolves into a disordered (incoherent) state.
Magnitude of order parameter vs. time

$K = 0.10$, $N = 256$, $\sigma = 0.2$
Strong Coupling

For large values of $K$ the system becomes orderly but not all oscillators synchronize.
The system levels off with a value of $r = r_\infty < 1$.
The system is only partially synchronized.
Magnitude of order parameter vs. time

$K = 0.50$, $N = 256$, $\sigma = 0.2$
In the partially synchronized case the oscillators are in two groups. The first group has natural frequencies $\omega_i$ within $2\sigma$ of $\bar{\omega}$. These have synchronized at $\bar{\omega}$ and rotate with the average phase $\phi$. The second group has $\omega_i$ outside $2\sigma$ of $\bar{\omega}$. These run near their natural frequencies and drift relative to the cluster of synchronized oscillators.
$K = 2.00$
$N = 32$
$t = 40.000$

angle represents phase $\theta_i$

color represents frequency $\omega_i - \omega_{mean}$

$2s$

$s$

$0$

$-s$

$-2s$
We can plot the value of \( r_\infty \) as a function of \( K \).

For values of \( K \) less than some critical value \( K_c \) the system stays disordered.

But for \( K > K_c \) the system becomes ordered.

This looks like a phase transition.
Kuramoto’s Analysis

Kuramoto was able to get formulas for $K_c$ and $r_\infty$.

$$K_c = \frac{2}{\pi g(0)}$$

This is an exact formula for any $g$ satisfying the conditions above.

For the special case of

$$g(\omega) = \frac{\gamma}{\pi (\gamma^2 + \omega^2)}$$

Kuramoto was able to get

$$r_\infty = \sqrt{1 - \frac{K_c}{K}}$$
Stability

Kuramoto and Nishikawa tried to prove that the system was stable. They published two papers in 1987. Neither turned out to be correct.

Strogatz and Mirollo (1991) and Crawford (1995) proved that the incoherent state is stable.

But as of 2000, nobody has been able to prove that the ordered state is stable.


