The Jacobi polynomials, their Sobolev orthogonality, and self-adjoint operators

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It is well known that, for $-\alpha, -\beta, -\alpha-\beta-1 \notin \mathbb{N}$, the Jacobi polynomials $\left\{P_n^{(\alpha,\beta)}(x)\right\}_{n=0}^{\infty}$ are orthogonal on $\mathbb{R}$ with respect to a bilinear form of the type $(f,g)_\mu = \int_{\mathbb{R}} fg d\mu$ for some measure μ. However, for negative integer parameters α and β, an application of Favard's theorem shows that the Jacobi polynomials cannot be orthogonal on the real line with respect to a bilinear form of this type for *any* positive or signed measure. But it is known that they are orthogonal with respect to a Sobolev inner product. Here, we will consider the special case where $\alpha = \beta = -1$. I will talk about the Sobolev orthogonality of the Jacobi polynomials and construct a self-adjoint operator in a certain Hilbert-Sobolev space having the entire sequence of Jacobi polynomials as eigenfunctions. The key to this construction is the left-definite spectral theory associated with the Jacobi differential equation, and the left-definite spaces and operators will be constructed explicitly. These results generalize to the case where $\alpha > -1, \beta = -1$.