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Math Colloquium
Thursday
March 21, 2013
UC 307
12:30 p.m.
(Refreshments at 12:15pm)



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The Linear Diophantine Frobenius Problem: An Elementary Introduction to Numerical Monoids

ABSTRACT: How many sums are NOT payable using 5p(enny) and 12p coins? What's the largest such sum? How many ways can a sum be paid? If McNuggets come in boxes of 6, 9 and 20, which amounts cannot be purchased? What's the largest such amount? If in a certain game one can score 100 or 101 or 110 or 111 points at each round, how many total points are impossible? What's the largest one?

In the puzzles above we are asked to find the genus and the conductor (or the Frobenius number) of a numerical monoid (i.e., a cofinite subset of non-negative integers containing 0, closed under addition). These elementary problems in combinatorial number theory are intimately connected with homological and commutative algebra (graded resolutions, Gorenstein rings), algebraic geometry (monomial curves, Arf rings), Convexity (lattice points in polyhedra), Harmonic analysis (Dedekind sums), complexity theory, operations research, symmetries of surfaces, etc. Several recent books (by Alfonsin-Ramirez, Rosales & Garcia-Sanchez, Leher) are about the algebraic aspects this problem, but there are still more questions than answers. On the analysis side there is some recent progress on Arnold's conjectures on the limiting behaviour of "average" Frobenius numbers.

The question of computing the genus (the size of the complement of the numerical monoid) for two (relatively prime) generators a and b was solved by J. J. Sylvester in the 1882: $\text{genus} = (a-1)(b-1)/2$. The answer for three or more generators can not be an algebraic function of the generators (and it took another century to figure it out). With four or more generators little is known except for very special cases (like arithmetic or geometric progressions). Finding the conductor (the largest element of the complement) of the numerical monoid is known as the linear Diophantine Frobenius problem, again the best results beyond three generators (in the general case) are about polynomial-time algorithms.

The Hilbert series (of the numerical monoid), expressed as a "short" rational function is an effective method for computing both the genus and the conductor. The talk will mostly be about explaining, finding and using the Hilbert series, illustrated with examples. I will try to sketch an elementary proof (that could have been given in Sylvester's time) of the solution for three coins. A satisfactory explicit answer to the third question "How many ways ..." is only known for two generators (Popoviciu's Theorem). The only known general formula(s) for 3 or more generators involves Fourier-Dedekind sums.