Interlacing of Zeroes and Wendroff’s Theorem

Kathy Driver is Professor of Mathematics at the University of Cape Town, South Africa. She was Professor and Chair of the Mathematics Department at Witwatersrand University, Johannesburg from 1998 to 2005 and Dean of the Faculty of Science at the University of Cape Town from 2006 to 2010.

Her research area is Orthogonal Polynomials and Special Functions and she has published more than 75 papers in international journals. She is an Associate Editor of the Journal of Mathematical Analysis and Applications.

Abstract. Suppose \( \{x_i\}_{i=1}^{n} \) and \( \{y_i\}_{i=1}^{n-1} \) are two sets of real, distinct points satisfying the interlacing property \( x_1 < y_1 < x_2 < y_2 < \cdots < x_{n-1} < y_{n-1} < x_n \). In 1961, Wendroff proved that if \( P_n(x) = \prod_{k=1}^{n} (x - x_k) \) and \( P_{n-1}(x) = \prod_{k=1}^{n-1} (x - y_k) \) are the monic (highest coefficient equal to 1) polynomials with simple zeros at \( x_1, x_2, \ldots, x_n \) and \( y_1, y_2, \ldots, y_{n-1} \) respectively, there exist infinitely many sequences \( \{P_k\}_{k=0}^{\infty} \) of orthogonal polynomials with \( P_n(x) = \prod_{k=1}^{n} (x - x_k) \) and \( P_{n-1}(x) = \prod_{k=1}^{n-1} (x - y_k) \). We explain what sequences of orthogonal polynomials are and why they are important and we give the original straightforward beautiful elegant proof of Wendroff’s Theorem which shows that the interlacing of the zeros of \( P_n \) and \( P_{n-1} \) is crucial in the construction of orthogonal sequences.