Chapter 22
The Electric Field 2: Continuous Charge Distributions

Conceptual Problems

1

Figure 22-37 shows an L-shaped object that has sides which are equal in length. Positive charge is distributed uniformly along the length of the object. What is the direction of the electric field along the dashed 45° line? Explain your answer.

Determine the Concept The resultant field is the superposition of the electric fields due to the charge distributions along the axes and is directed along the dashed line, pointing away from the intersection of the two sides of the L-shaped object. This can be seen by dividing each leg of the object into 10 (or more) equal segments and then drawing the electric field on the dashed line due to the charges on each pair of segments that are equidistant from the intersection of the legs.

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An electric dipole is completely inside a closed imaginary surface and there are no other charges. True or False:

(a) The electric field is zero everywhere on the surface.
(b) The electric field is normal to the surface everywhere on the surface.
(c) The electric flux through the surface is zero.
(d) The electric flux through the surface could be positive or negative.
(e) The electric flux through a portion of the surface might not be zero.

(a) False. Near the positive end of the dipole, the electric field, in accordance with Coulomb’s law, will be directed outward and will be nonzero. Near the negative end of the dipole, the electric field, in accordance with Coulomb’s law, will be directed inward and will be nonzero.

(b) False. The electric field is perpendicular to the Gaussian surface only at the intersections of the surface with a line defined by the axis of the dipole.

(c) True. Because the net charge enclosed by the Gaussian surface is zero, the net flux, given by \( \phi_{\text{net}} = \oint_S E_s dA = 4\pi k Q_{\text{inside}} \), through this surface must be zero.

(d) False. The flux through the closed surface is zero.

(e) True. All Gauss’s law tells us is that, because the net charge inside the surface is zero, the net flux through the surface must be zero.
Suppose that the total charge on the conducting spherical shell in Figure 22-38 is zero. The negative point charge at the center has a magnitude given by \( Q \). What is the direction of the electric field in the following regions? 
(a) \( r < R_1 \), (b) \( R_2 > r > R_1 \), (c) and \( r > R_2 \). Explain your answer.

**Determine the Concept** We can apply Gauss’s law to determine the electric field for \( r < R_1 \), \( R_2 > r > R_1 \), and \( r > R_2 \). We also know that the direction of an electric field at any point is determined by the direction of the electric force acting on a positively charged object located at that point.

(a) From the application of Gauss’s law we know that the electric field in this region is not zero. A positively charged object placed in the region for which \( r < R_1 \) will experience an attractive force from the charge \(-Q\) located at the center of the shell. Hence the direction of the electric field is radially inward.

(b) Because the total charge on the conducting sphere is zero, the charge on its inner surface is \(+Q\) (the positive charges in the conducting sphere are drawn there by the negative charge at the center of the shell) and the charge on its outer surface is \(-Q\). Applying Gauss’s law in the region \( R_2 > r > R_1 \) (the net charge enclosed by a Gaussian surface of radius \( r \) is zero) leads to the conclusion that the electric field in this region is zero. It has no direction.

(c) Because the charge on the outer surface of the conducting shell is negative, the electric field in the region \( r > R_2 \) is radially inward.

**Calculating \( \vec{E} \) From Coulomb’s Law**

A uniform line charge that has a linear charge density \( \lambda \) equal to 3.5 nC/m is on the \( x \) axis between \( x = 0 \) and \( x = 5.0 \) m. (a) What is its total charge? Find the electric field on the \( x \) axis at \( (b) x = 6.0 \m, (c) x = 9.0 \m, and (d) x = 250 \m \). (e) Estimate the electric field at \( x = 250 \m \), using the approximation that the charge is a point charge on the \( x \) axis at \( x = 2.5 \m \), and compare your result with the result calculated in Part (d). (To do this you will need to assume that the values given in this problem statement are valid to more than two significant figures.) Is your approximate result greater or smaller than the exact result? Explain your answer.

**Picture the Problem** (a) We can use the definition of \( \lambda \) to find the total charge of the line of charge. (b), (c) and (d) Equation 22-2b gives the electric field on the axis of a finite line of charge. In Part (e) we can apply Coulomb’s law for the electric field due to a point charge to approximate the electric field at \( x = 250 \m \). In the following diagram, \( L = 5.0 \m \) and \( P \) is a generic point on the \( x \) axis.
The electric field on the axis of a finite line charge is given by Equation 22-2b:

\[ E_x = k \lambda \left( \frac{1}{r_2} - \frac{1}{r_1} \right) \]

(a) Use the definition of linear charge density to express \( Q \) in terms of \( \lambda \):

\[ Q = \lambda L = (3.5 \text{nC/m})(5.0 \text{m}) = 17.5 \text{nC} = 18 \text{nC} \]

(b) Substitute numerical values and evaluate \( E_x \) at \( x = 6.0 \text{ m} \):

\[ E_{x-6.0 \text{ m}} = \left(8.988 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}\right) \left(3.5 \times 10^{-9} \frac{\text{C}}{\text{m}}\right) \left(\frac{1}{6.0 \text{ m} - 5.0 \text{ m}} - \frac{1}{6.0 \text{ m}}\right) = 26 \text{ N/C} \]

(c) Substitute numerical values and evaluate \( E_x \) at \( x = 9.0 \text{ m} \):

\[ E_{x-9.0 \text{ m}} = \left(8.988 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}\right) \left(3.5 \times 10^{-9} \frac{\text{C}}{\text{m}}\right) \left(\frac{1}{9.0 \text{ m} - 5.0 \text{ m}} - \frac{1}{9.0 \text{ m}}\right) = 4.4 \text{ N/C} \]

(d) Substitute numerical values and evaluate \( E_x \) at \( x = 250 \text{ m} \):

\[ E_{x-250 \text{ m}} = \left(8.988 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}\right) \left(3.5 \times 10^{-9} \frac{\text{C}}{\text{m}}\right) \left(\frac{1}{250 \text{ m} - 5.0 \text{ m}} - \frac{1}{250 \text{ m}}\right) = 2.56800 \text{ mN/C} = 2.6 \text{ mN/C} \]

(e) Using the approximation that the charge is a point charge on the \( x \) axis at \( x = 2.5 \text{ m} \), Coulomb’s law gives:

\[ E_x = \frac{kQ}{(r_1 - \frac{1}{2} L)^2} \]

Substitute numerical values and evaluate \( E_x \) at \( x = 250 \text{ m} \):

\[ E_{x-250 \text{ m}} = \left(8.988 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}\right)(17.5 \text{nC}) \left(250 \text{ m} - \frac{1}{2}(5.0 \text{ m})\right)^2 = 2.56774 \text{ mN/C} = 2.6 \text{ mN/C} \]

This result is about 0.01% less than the exact value obtained in (d). This suggests that the line of charge can be modeled to within 0.01% as that due to a point charge.
A ring that has radius \( a \) lies in the \( z = 0 \) plane with its center at the origin. The ring is uniformly charged and has a total charge \( Q \). Find \( E_z \) on the \( z \)-axis at (a) \( z = 0.2a \), (b) \( z = 0.5a \), (c) \( z = 0.7a \), (d) \( z = a \), and (e) \( z = 2a \). (f) Use your results to plot \( E_z \) versus \( z \) for both positive and negative values of \( z \). (Assume that these distances are exact.)

**Picture the Problem** The electric field at a distance \( z \) from the center of a ring whose charge is \( Q \) and whose radius is \( a \) is given by 

\[
E_z = \frac{kQz}{(z^2 + a^2)^{3/2}}
\]

(a) Evaluating \( E_z = 0.2a \) gives:

\[
E_{z=0.2a} = \frac{kQ(0.2a)}{(0.2a)^2 + a^2} = 0.189 \frac{kQ}{a^2}
\]

(b) Evaluating \( E_z = 0.5a \) gives:

\[
E_{z=0.5a} = \frac{kQ(0.5a)}{(0.5a)^2 + a^2} = 0.358 \frac{kQ}{a^2}
\]

(c) Evaluating \( E_z = 0.7a \) gives:

\[
E_{z=0.7a} = \frac{kQ(0.7a)}{(0.7a)^2 + a^2} = 0.385 \frac{kQ}{a^2}
\]

(d) Evaluating \( E_z = a \) gives:

\[
E_{z=a} = \frac{kQa}{a^2 + a^2} = 0.354 \frac{kQ}{a^2}
\]

(e) Evaluating \( E_z = 2a \) gives:

\[
E_{z=2a} = \frac{2kQa}{(2a)^2 + a^2} = 0.179 \frac{kQ}{a^2}
\]

(f) The field along the \( z \) axis is plotted below. The \( z \) coordinates are in units of \( z/a \) and \( E \) is in units of \( kQ/a^2 \). 

![Graph of Electric Field vs. z/A](image-url)
25 Calculate the electric field a distance $z$ from a uniformly charged infinite flat non-conducting sheet by modeling the sheet as a continuum of infinite circular rings of charge.

**Picture the Problem** The field at a point on the axis of a uniformly charged ring lies along the axis and is given by Equation 22-8. The diagram shows one ring of the continuum of circular rings of charge. The radius of the ring is $a$ and the distance from its center to the field point $P$ is $z$. The ring has a uniformly distributed charge $Q$. The resultant electric field at $P$ is the sum of the fields due to the continuum of circular rings. Note that, by symmetry, the horizontal components of the electric field cancel.

Express the field of a single uniformly charged ring with charge $Q$ and radius $a$ on the axis of the ring at a distance $z$ away from the plane of the ring:

$$E = E_z \hat{i}, \text{ where } E_z = \frac{kQz}{(z^2 + a^2)^{3/2}}$$

Substitute $dq$ for $Q$ and $dE_z$ for $E_z$ to obtain:

$$dE_z = \frac{kz dq}{(z^2 + a^2)^{3/2}}$$

The resultant electric field at $P$ is the sum of the fields due to all the circular rings. Integrate both sides to calculate the resultant field for the entire plane. The field point remains fixed, so $z$ is constant:

$$E = \int \frac{kz dq}{(z^2 + a^2)^{3/2}} = k \int \frac{dq}{(z^2 + a^2)^{3/2}}$$

To evaluate this integral we change integration variables from $q$ to $a$. The charge $dq = \sigma dA$ where $dA = 2\pi a da$ is the area of a ring of radius $a$ and width $da$.

$$dq = 2\pi \sigma a da$$

so

$$E = k z \int_0^\infty \frac{2\pi \sigma a da}{\left( z^2 + a^2 \right)^{3/2}}$$

$$= 2\pi \sigma k z \int_0^\infty \frac{a da}{\left( z^2 + a^2 \right)^{3/2}}$$
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To integrate this expression, let

\[ u = \sqrt{z^2 + a^2} \]

Then:

\[ du = \frac{1}{2} \frac{1}{\sqrt{z^2 + a^2}} (2ada) = \frac{ada}{u} \]

or

\[ ada = udu \]

Noting that when \( a = 0 \), \( u = z \),

substitute and simplify to obtain:

\[ E = 2\pi\sigma kz \int_{u}^{\infty} \frac{u}{u^2} du = 2\pi\sigma kz \int_{u}^{\infty} u^{-2} du \]

Evaluating the integral yields:

\[ E = 2\pi\sigma kz \left( -\frac{1}{u} \right) \bigg|_{u}^{\infty} = 2\pi k \sigma \left( -\frac{1}{u} \right) \bigg|_{z}^{\infty} = 2\pi k \sigma \left( -\frac{1}{z} \right) \]

**Gauss’s Law**

29 • An electric field is given by \( \vec{E} = \text{sign}(x) \cdot (300 \text{ N/C}) \hat{i} \), where \( \text{sign}(x) \) equals \(-1 \) if \( x < 0 \), \( 0 \) if \( x = 0 \), and \(+1 \) if \( x > 0 \). A cylinder of length 20 cm and radius 4.0 cm has its center at the origin and its axis along the \( x \) axis such that one end is at \( x = +10 \) cm and the other is at \( x = -10 \) cm. (a) What is the electric flux through each end? (b) What is the electric flux through the curved surface of the cylinder? (c) What is the electric flux through the entire closed surface? (d) What is the net charge inside the cylinder?

**Picture the Problem** The field at both circular faces of the cylinder is parallel to the outward vector normal to the surface, so the flux is just \( EA \). There is no flux through the curved surface because the normal to that surface is perpendicular to \( \vec{E} \). The net flux through the closed surface is related to the net charge inside by Gauss’s law.
(a) Use Gauss’s law to calculate the flux through the right circular surface:
\[ \phi_{\text{right}} = \vec{E}_{\text{right}} \cdot \hat{n}_{\text{right}} A = (300 \text{ N/C}) \hat{i} \cdot \hat{i} (\pi) (0.040 \text{ m})^2 = 1.5 \text{ N} \cdot \text{m}^2/\text{C} \]

Apply Gauss’s law to the left circular surface:
\[ \phi_{\text{left}} = \vec{E}_{\text{left}} \cdot \hat{n}_{\text{left}} A = (-300 \text{ N/C}) \hat{i} \cdot (-\hat{i}) (\pi) (0.040 \text{ m})^2 = 1.5 \text{ N} \cdot \text{m}^2/\text{C} \]

(b) Because the field lines are parallel to the curved surface of the cylinder:
\[ \phi_{\text{curved}} = 0 \]

(c) Express and evaluate the net flux through the entire cylindrical surface:
\[ \phi_{\text{net}} = \phi_{\text{right}} + \phi_{\text{left}} + \phi_{\text{curved}} = 1.5 \text{ N} \cdot \text{m}^2/\text{C} + 1.5 \text{ N} \cdot \text{m}^2/\text{C} + 0 = 3.0 \text{ N} \cdot \text{m}^2/\text{C} \]

(d) Apply Gauss’s law to obtain:
\[ \phi_{\text{net}} = 4\pi k Q_{\text{inside}} \Rightarrow Q_{\text{inside}} = \frac{\phi_{\text{net}}}{4\pi k} \]

Substitute numerical values and evaluate \( Q_{\text{inside}} \):
\[ Q_{\text{inside}} = \frac{3.0 \text{ N} \cdot \text{m}^2/\text{C}}{4\pi (8.988 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)} = 2.7 \times 10^{-11} \text{ C} \]

33 • A single point charge is placed at the center of an imaginary cube that has 20-cm-long edges. The electric flux out of one of the cube's sides is \(-1.50 \text{ kN} \cdot \text{m}^2/\text{C}\). How much charge is at the center?

**Picture the Problem** The net flux through the cube is given by \( \phi_{\text{net}} = \frac{Q_{\text{inside}}}{\varepsilon_0} \), where \( Q_{\text{inside}} \) is the charge at the center of the cube.

The flux through one side of the cube is one-sixth of the total flux through the cube:
\[ \phi_{\text{faces}} = \frac{1}{6} \phi_{\text{net}} = \frac{Q_{\text{inside}}}{6 \varepsilon_0} \]

Solving for \( Q_{\text{inside}} \) yields:
\[ Q_{\text{inside}} = 6 \varepsilon_0 \phi_{\text{faces}} \]
Substitute numerical values and evaluate $Q_{\text{inside}}$:

$$Q_{\text{inside}} = 6 \left( 8.854 \times 10^{-12} \frac{\text{C}^2}{\text{N} \cdot \text{m}^2} \right) \left( -1.50 \frac{\text{kN} \cdot \text{m}^2}{\text{C}^2} \right) = -79.7 \text{ nC}$$

**Gauss’s Law Applications in Spherical Symmetry Situations**

39 ** A non-conducting sphere of radius 6.00 cm has a uniform volume charge density of 450 nC/m$^3$. (a) What is the total charge on the sphere? Find the electric field at the following distances from the sphere’s center: (b) 2.00 cm, (c) 5.90 cm, (d) 6.10 cm, and (e) 10.0 cm.

(a) Using the definition of volume charge density, relate the charge on the sphere to its volume:

$$Q = \rho V = \frac{4}{3} \pi \rho r^3$$

Substitute numerical values and evaluate $Q$:

$$Q = \frac{4}{3} \pi (450 \text{nC/m}^3)(0.0600 \text{m})^3$$

$$= 0.4072 \text{nC} = 0.407 \text{nC}$$

Apply Gauss’s law to a spherical surface of radius $r < R$ that is concentric with the spherical shell to obtain:

$$\oint_S E_n dA = \frac{1}{\varepsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_n = \frac{Q_{\text{inside}}}{\varepsilon_0}$$

Noting that, due to symmetry, $E_n = E_r$, solve for $E_r$ to obtain:

$$E_r = \frac{Q_{\text{inside}}}{4\pi \varepsilon_0} \frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2}$$

Because the charge distribution is uniform, we can find the charge inside the Gaussian surface by using the definition of volume charge density to establish the proportion:

$$\frac{Q}{V} = \frac{Q'}{V'}$$

where $V'$ is the volume of the Gaussian surface.

Solve for $Q_{\text{inside}}$ to obtain:

$$Q_{\text{inside}} = \frac{Q V'}{V} = \frac{Q r^3}{R^3}$$

Substitute for $Q_{\text{inside}}$ to obtain:

$$E_{r < R} = \frac{Q_{\text{inside}}}{4\pi \varepsilon_0} \frac{1}{r^2} = \frac{kQ}{R^3} r$$
(b) Evaluate $E_r = 2.00 \text{ cm}$:

$$E_{r=2.00\text{ cm}} = \frac{\left(8.988 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2\right)(0.4072 \text{nC})(0.0200 \text{ m})}{(0.0600 \text{ m})^3} = 339 \text{ N/C}$$

(c) Evaluate $E_r = 5.90 \text{ cm}$:

$$E_{r=5.90\text{ cm}} = \frac{\left(8.988 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2\right)(0.4072 \text{nC})(0.0590 \text{ m})}{(0.0600 \text{ m})^3} = 1.00 \text{ kN/C}$$

Apply Gauss’s law to the Gaussian surface with $r > R$:

$$4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\varepsilon_0} \Rightarrow E_r = k\frac{Q_{\text{inside}}}{r^2} = k\frac{Q}{r^2}$$

(d) Evaluate $E_r = 6.10 \text{ cm}$:

$$E_{r=6.10\text{ cm}} = \frac{\left(8.988 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2\right)(0.4072 \text{nC})}{(0.0610 \text{ m})^3} = 983 \text{ N/C}$$

(e) Evaluate $E_r = 10.0 \text{ cm}$:

$$E_{r=10.0\text{ cm}} = \frac{\left(8.988 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2\right)(0.4072 \text{nC})}{(0.100 \text{ m})^3} = 366 \text{ N/C}$$

43 A sphere of radius $R$ has volume charge density $\rho = B/r$ for $r < R$, where $B$ is a constant and $\rho = 0$ for $r > R$. (a) Find the total charge on the sphere. (b) Find the expressions for the electric field inside and outside the charge distribution (c) Sketch the magnitude of the electric field as a function of the distance $r$ from the sphere’s center.

**Picture the Problem** We can find the total charge on the sphere by expressing the charge $dq$ in a spherical shell and integrating this expression between $r = 0$ and $r = R$. By symmetry, the electric fields must be radial. To find $E_r$ inside the charged sphere we choose a spherical Gaussian surface of radius $r < R$. To find $E_r$ outside the charged sphere we choose a spherical Gaussian surface of radius $r > R$. On each of these surfaces, $E_r$ is constant. Gauss’s law then relates $E_r$ to the total charge inside the surface.

(a) Express the charge $dq$ in a shell of thickness $dr$ and volume $4\pi r^2 \text{ dr}$:

$$dq = 4\pi r^2 \rho dr = 4\pi \frac{B}{r^2} \frac{B}{r} dr$$

$$= 4\pi B r^2 dr$$
Integrate this expression from $r = 0$ to $R$ to find the total charge on the sphere:

$$Q = 4\pi B \int_0^R r^2 dr = \left[2\pi Br^2\right]_0^R = 2\pi BR^2$$

(b) Apply Gauss’s law to a spherical surface of radius $r > R$ that is concentric with the nonconducting sphere to obtain:

$$\int_S E_r dA = \frac{1}{\varepsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\varepsilon_0}$$

Solving for $E_r$ yields:

$$E_{r>R} = \frac{Q_{\text{inside}}}{4\pi \varepsilon_0} \frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2} = \frac{k2\pi BR^2}{r^2} = \frac{BR^2}{2 \varepsilon_0 r^2}$$

Apply Gauss’s law to a spherical surface of radius $r < R$ that is concentric with the nonconducting sphere to obtain:

$$\int_S E_r dA = \frac{1}{\varepsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\varepsilon_0}$$

Solving for $E_r$ yields:

$$E_{r<R} = \frac{Q_{\text{inside}}}{4\pi \varepsilon_0} = \frac{2\pi Br^2}{4\pi \varepsilon_0} = \frac{B}{2 \varepsilon_0}$$

(c) The following graph of $E_r$ versus $r/R$, with $E_r$ in units of $B/(2 \varepsilon_0)$, was plotted using a spreadsheet program.

![Graph of E_r vs. r/R](image)

Remarks: Note that our results for (a) and (b) agree at $r = R$. 
**Gauss’s Law Applications in Cylindrical Symmetry Situations**

51  
A solid cylinder of length 200 m and radius 6.00 cm has a uniform volume charge density of 300 nC/m$^3$. (a) What is the total charge of the cylinder? Use the formulas given in Problem 50 to calculate the electric field at a point equidistant from the ends at the following radial distances from the cylindrical axis: (b) 2.00 cm, (c) 5.90 cm, (d) 6.10 cm, and (e) 10.0 cm.

**Picture the Problem**  
We can use the definition of volume charge density to find the total charge on the cylinder. From symmetry, the electric field tangent to the surface of the cylinder must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius $r$ and length $L$ and apply Gauss’s law to find the electric field as a function of the distance from the centerline of the uniformly charged cylinder.

(a) Use the definition of volume charge density to express the total charge of the cylinder:

$$Q_{\text{total}} = \rho V = \rho (\pi R^2 L)$$

Substitute numerical values to obtain:

$$Q_{\text{total}} = \pi (300 \text{ nC/m}^3)(0.0600 \text{ m})^2 (200 \text{ m}) = 679 \text{ nC}$$

(b) From Problem 50, for $r < R$, we have:

For $r = 2.00$ cm:

$$E_{r<R} = \frac{\rho}{2 \varepsilon_0} r$$

$$E_{r=2.00 \text{ cm}} = \frac{(300 \text{ nC/m}^3)(0.0200 \text{ m})}{2(8.854 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)} = 339 \text{ N/C}$$

(c) For $r = 5.90$ cm:

$$E_{r=5.90 \text{ cm}} = \frac{(300 \text{ nC/m}^3)(0.0590 \text{ m})}{2(8.854 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)} = 1.00 \text{ kN/C}$$

From Problem 50, for $r > R$, we have:

$$E_{r>R} = \frac{\rho R^2}{2 \varepsilon_0 r}$$
(d) For \( r = 6.10 \text{ cm} \):

\[
E_{r=6.10 \text{ cm}} = \frac{(300 \text{nC/m}^3)(0.0600 \text{ m})^2}{2(8.854 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)(0.0610 \text{ m})} = 1.00 \text{kN/C}
\]

(e) For \( r = 10.0 \text{ cm} \):

\[
E_{r=10.0 \text{ cm}} = \frac{(300 \text{nC/m}^3)(0.0600 \text{ m})^2}{2(8.854 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)(0.100 \text{ m})} = 610 \text{ N/C}
\]

55 ** An infinitely long non-conducting solid cylinder of radius \( a \) has a non-uniform volume charge density. This density varies with \( R \), the perpendicular distance from its axis, according to \( \rho(R) = bR^2 \), where \( b \) is a constant. (a) Show that the linear charge density of the cylinder is given by \( \lambda = \pi ba^4/2 \). (b) Find expressions for the electric field for \( R < a \) and \( R > a \).

**Picture the Problem** From symmetry; the field tangent to the surface of the cylinder must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius \( R \) and length \( L \) and apply Gauss’s law to find the electric field as a function of the distance from the centerline of the infinitely long nonconducting cylinder.

(a) Apply Gauss’s law to a cylindrical surface of radius \( R \) and length \( L \) that is concentric with the infinitely long nonconducting cylinder:

\[
\oint_S E_n \, dA = \frac{1}{\varepsilon_0} \int_{\text{inside}} Q \Rightarrow 2\pi RE_n = \frac{Q_{\text{inside}}}{\varepsilon_0}
\]

where we’ve neglected the end areas because there is no flux through them.

Noting that, due to symmetry, \( E_n = E_R \), solve for \( E_R \) to obtain:

\[
E_R = \frac{Q_{\text{inside}}}{2\pi RL \varepsilon_0}
\]

(1)

Express \( dQ_{\text{inside}} \) for \( \rho(R) = bR^2 \):

\[
dQ_{\text{inside}} = \rho(R) \, dV = bR^2 (2\pi RL) \, dR
\]

Integrate \( dQ_{\text{inside}} \) from 0 to \( a \) to obtain:

\[
Q_{\text{inside}} = 2\pi bL \int_0^a R^3 \, dR = 2\pi bL \left[ \frac{R^4}{4} \right]_0^a = \frac{\pi ba^4}{2}
\]
Divide both sides of this equation by $L$ to obtain an expression for the charge per unit length $\lambda$ of the cylinder:

\[ \lambda = \frac{Q_{\text{inside}}}{L} = \frac{\pi b a^4}{2} \]

(b) Substitute for $Q_{\text{inside}}$ in equation (1) and simplify to obtain:

\[ E_{R=a} = \frac{\pi b L}{2} a^4 \]

For $R > a$:

\[ Q_{\text{inside}} = \frac{\pi b L}{2} a^4 \]

Substitute for $Q_{\text{inside}}$ in equation (1) and simplify to obtain:

\[ E_{R=a} = \frac{\pi b L}{2} a^4 R \]

57 57 57 The inner cylinder of Figure 22-42 is made of non-conducting material and has a volume charge distribution given by $\rho(R) = C/R$, where $C = 200$ nC/m$^2$. The outer cylinder is metallic, and both cylinders are infinitely long. (a) Find the charge per unit length (that is, the linear charge density) on the inner cylinder. (b) Calculate the electric field for all values of $R$.

Picture the Problem We can integrate the density function over the radius of the inner cylinder to find the charge on it and then calculate the linear charge density from its definition. To find the electric field for all values of $r$ we can construct a Gaussian surface in the shape of a cylinder of radius $R$ and length $L$ and apply Gauss’s law to each region of the cable to find the electric field as a function of the distance from its centerline.

(a) Letting the radius of the inner cylinder be $a$, find the charge $Q_{\text{inner}}$ on the inner cylinder:

\[ Q_{\text{inner}} = \int_0^a \rho(R) dV = \int_0^a C \frac{2\pi R L dR}{R} = 2\pi C L a \]

Relate this charge to the linear charge density:

\[ \lambda = \frac{Q_{\text{inner}}}{L} = \frac{2\pi C L a}{L} = 2\pi C a \]

Substitute numerical values and evaluate $\lambda$:

\[ \lambda = 2\pi (200 \text{nC/m}) (0.0150 \text{m}) = 18.8 \text{nC/m} \]
(b) Apply Gauss’s law to a cylindrical surface of radius $r$ and length $L$ that is concentric with the infinitely long nonconducting cylinder:

$$\oint_{S} E_{n} \, dA = \frac{1}{\varepsilon_{0}} Q_{\text{inside}} \Rightarrow 2\pi r L E_{n} = \frac{Q_{\text{inside}}}{\varepsilon_{0}}$$

where we’ve neglected the end areas because there is no flux through them.

Noting that, due to symmetry, $E_{n} = E_{R}$, solve for $E_{R}$ to obtain:

$$E_{R} = \frac{Q_{\text{inside}}}{2\pi r L \varepsilon_{0}}$$

Substitute to obtain, for $R < 1.50$ cm:

$$E_{R < 1.50 \text{ cm}} = \frac{2\pi C L R}{2\pi r L R} = \frac{C}{\varepsilon_{0}}$$

Substitute numerical values and evaluate $E_{n}(R < 1.50 \text{ cm})$:

$$E_{R < 1.50 \text{ cm}} = \frac{200 \text{nC/m}^2}{8.854 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2} = 22.6 \text{kN/C}$$

Express $Q_{\text{inside}}$ for $1.50 \text{ cm} < R < 4.50 \text{ cm}$:

$$Q_{\text{inside}} = 2\pi CLa$$

Substitute to obtain, for $1.50 \text{ cm} < R < 4.50 \text{ cm}$:

$$E_{R > 1.50 \text{ cm} < 4.50 \text{ cm}} = \frac{2C a L}{2\pi \varepsilon_{0} R L} = \frac{Ca}{\varepsilon_{0} R}$$

where $R = 1.50$ cm.

Substitute numerical values and evaluate $E_{R > 1.50 \text{ cm} < 4.50 \text{ cm}}$:

$$E_{R > 1.50 \text{ cm} < 4.50 \text{ cm}} = \frac{(200 \text{nC/m}^2)(0.0150 \text{ m})}{(8.854 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)} = \frac{339 \text{ N} \cdot \text{m/C}}{R}$$

Because the outer cylindrical shell is a conductor:

$$E_{R > 6.50 \text{ cm}} = 0$$

For $R > 6.50$ cm, $Q_{\text{inside}} = 2\pi CL$ and:

$$E_{R > 6.50 \text{ cm}} = \frac{339 \text{ N} \cdot \text{m/C}}{R}$$

**Electric Charge and Field at Conductor Surfaces**

A positive point charge of $2.5 \mu\text{C}$ is at the center of a conducting spherical shell that has a net charge of zero, an inner radius equal to 60 cm, and an outer radius equal to 90 cm. (a) Find the charge densities on the inner and outer surfaces of the shell and the total charge on each surface. (b) Find the electric
field everywhere. (c) Repeat Part (a) and Part (b) with a net charge of +3.5 $\mu$C placed on the shell.

**Picture the Problem** Let the inner and outer radii of the uncharged spherical conducting shell be $R_1$ and $R_2$ and $q$ represent the positive point charge at the center of the shell. The positive point charge at the center will induce a negative charge on the inner surface of the shell and, because the shell is uncharged, an equal positive charge will be induced on its outer surface. To solve Part (b), we can construct a Gaussian surface in the shape of a sphere of radius $r$ with the same center as the shell and apply Gauss’s law to find the electric field as a function of the distance from this point. In Part (c) we can use a similar strategy with the additional charge placed on the shell.

(a) Express the charge density on the inner surface:
\[
\sigma_{\text{inner}} = \frac{q_{\text{inner}}}{A}
\]
Express the relationship between the positive point charge $q$ and the charge induced on the inner surface $q_{\text{inner}}$:
\[
 q + q_{\text{inner}} = 0 \Rightarrow q_{\text{inner}} = -q
\]
Substitute for $q_{\text{inner}}$ and $A$ to obtain:
\[
\sigma_{\text{inner}} = \frac{-q}{4\pi R_1^2}
\]
Substitute numerical values and evaluate $\sigma_{\text{inner}}$:
\[
\sigma_{\text{inner}} = \frac{-2.5 \, \mu C}{4\pi (0.60 \, \text{m})^2} = -0.55 \, \mu \text{C/m}^2
\]
Express the charge density on the outer surface:
\[
\sigma_{\text{outer}} = \frac{q_{\text{outer}}}{A}
\]
Because the spherical shell is uncharged:
\[
 q_{\text{outer}} + q_{\text{inner}} = 0
\]
Substitute for $q_{\text{outer}}$ to obtain:
\[
\sigma_{\text{outer}} = \frac{-q_{\text{inner}}}{4\pi R_2^2}
\]
Substitute numerical values and evaluate $\sigma_{\text{outer}}$:
\[
\sigma_{\text{outer}} = \frac{2.5 \, \mu \text{C}}{4\pi (0.90 \, \text{m})^2} = 0.25 \, \mu \text{C/m}^2
\]
(b) Apply Gauss’s law to a spherical surface of radius \( r \) that is concentric with the point charge:

\[
\oint_S \mathbf{E}_n \, dA = \frac{1}{\varepsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_n = \frac{Q_{\text{inside}}}{\varepsilon_0}
\]

Noting that, due to symmetry, \( E_n = E_r \), solve for \( E_r \) to obtain:

\[
E_r = \frac{Q_{\text{inside}}}{4\pi r^2 \varepsilon_0} \quad (1)
\]

For \( r < R_1 = 60 \text{ cm} \), \( Q_{\text{inside}} = q \).
Substitute in equation (1) to obtain:

\[
E_{r<60 \text{ cm}} = \frac{q}{4\pi r^2 \varepsilon_0} = \frac{kq}{r^2}
\]

Substitute numerical values and evaluate \( E_{r<60 \text{ cm}} \):

\[
E_{r<60 \text{ cm}} = \left(\frac{8.988 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2}{r^2}\right) \left(2.5 \mu\text{C}\right) = \left(2.2 \times 10^4 \text{ N} \cdot \text{m}^2/\text{C}\right) \frac{1}{r^2}
\]

Because the spherical shell is a conductor, a charge \(-q\) will be induced on its inner surface. Hence, for \( 60 \text{ cm} < r < 90 \text{ cm} \):

\[
Q_{\text{inside}} = 0
\]

and

\[
E_{60 \text{ cm} < r < 90 \text{ cm}} = 0
\]

For \( r > 90 \text{ cm} \), the net charge inside the Gaussian surface is \( q \) and:

\[
E_{r>90 \text{ cm}} = \frac{kq}{r^2} = \left(\frac{2.3 \times 10^4 \text{ N} \cdot \text{m}^2/\text{C}}{r^2}\right)
\]

(c) Because \( E = 0 \) in the conductor:

\[
q_{\text{inner}} = -2.5 \mu\text{C}
\]

and

\[
\sigma_{\text{inner}} = -0.55 \mu\text{C/m}^2 \text{ as before.}
\]

Express the relationship between the charges on the inner and outer surfaces of the spherical shell:

\[
q_{\text{outer}} + q_{\text{inner}} = 3.5 \mu\text{C}
\]

and

\[
q_{\text{outer}} = 3.5 \mu\text{C} - q_{\text{inner}} = 6.0 \mu\text{C}
\]

\( \sigma_{\text{outer}} \) is now given by:

\[
\sigma_{\text{outer}} = \frac{6.0 \mu\text{C}}{4\pi (0.90 \text{ m})^2} = 0.59 \mu\text{C/m}^2
\]

For \( r < R_1 = 60 \text{ cm} \), \( Q_{\text{inside}} = q \) and \( E_{r<60 \text{ cm}} \) is as it was in (a):

\[
E_{r<60 \text{ cm}} = \left(\frac{2.3 \times 10^4 \text{ N} \cdot \text{m}^2/\text{C}}{r^2}\right)
\]
Because the spherical shell is a conductor, a charge \(-q\) will be induced on its inner surface. Hence, for \(60 \text{ cm} < r < 90 \text{ cm}\):

\[ Q_{\text{inside}} = 0 \quad \text{and} \quad E_{60 \text{ cm} < r < 90 \text{ cm}} = 0 \]

For \(r > 0.90 \text{ m}\), the net charge inside the Gaussian surface is 6.0 \(\mu\text{C}\) and:

\[
E_{r>90 \text{ cm}} = \frac{kq}{r^2} = \left(8.988 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2\right)\left(6.0 \mu\text{C}\right)\frac{1}{r^2} = \left(5.4 \times 10^4 \text{ N} \cdot \text{m}^2/\text{C}\right)\frac{1}{r^2}
\]

65  [SSM] A thin square conducting sheet that has 5.00-m-long edges has a net charge of 80.0 \(\mu\text{C}\). The square is in the \(x = 0\) plane and is centered at the origin. (Assume the charge on each surface is uniformly distributed.) (a) Find the charge density on each side of the sheet and find the electric field on the \(x\) axis in the region \(|x| < 5.00\) m. (b) A thin but infinite nonconducting sheet that has a uniform charge density of 2.00 \(\mu\text{C/m}^2\) is now placed in the \(x = -2.50\) m plane. Find the electric field on the \(x\) axis on each side of the square sheet in the region \(|x| < 2.50\) m. Find the charge density on each surface of the square sheet.

**Picture the Problem** (a) One half of the total charge is on each side of the square thin conducting sheet and the electric field inside the sheet is zero. The electric field intensity just outside the surface of a conductor is given by \(E = |\sigma|/\varepsilon_0\). Typical field points to the left and right of the square thin conducting sheet are shown in the following diagram.

\(Q_{\text{net}} = 80.0 \mu\text{C}\)

(b) We can use the fact that the net charge on the conducting sheet is the sum of the charges \(Q_{\text{left}}\) and \(Q_{\text{right}}\) on its left and right surfaces to obtain an equation relating these charges. Because the resultant electric field is zero inside the sheet, we can obtain a second equation in \(Q_{\text{left}}\) and \(Q_{\text{right}}\) that we can solve simultaneously with the first equation to find \(Q_{\text{left}}\) and \(Q_{\text{right}}\). The resultant electric field is the superposition of three fields—the field due to the charges on the infinite nonconducting sheet and the fields due to the charges on the surfaces of the thin square conducting sheet. The electric field intensity due to a uniformly charged
nonconducting infinite sheet is given by \( E = \frac{\sigma}{2 \varepsilon_0} \). Typical field points for each of the four regions of interest are shown in the following diagram.

Note: The vectors in this figure are drawn consistent with the charges \( Q_{\text{left}} \) and \( Q_{\text{right}} \) both being positive. If either \( Q_{\text{left}} \) or \( Q_{\text{right}} \) are negative then the solution will produce a negative value for either \( Q_{\text{left}} \) or \( Q_{\text{right}} \).

(a) Because the square sheet is a conductor, half the charge on each surface is half the net charge on the sheet:

\[
\sigma_{\text{left}} = \sigma_{\text{right}} = \frac{1}{2} \frac{Q_{\text{net}}}{A}
\]

Substitute numerical values and evaluate \( \sigma_{\text{left}} \) and \( \sigma_{\text{right}} \):

\[
\sigma_{\text{left}} = \sigma_{\text{right}} = \frac{1}{2} \left( \frac{80.0 \, \mu \text{C}}{5.00 \, \text{m}} \right)^2 = 1.60 \, \mu \text{C} \, \text{m}^{-2}
\]

For \( |x| < 5.00 \, \text{m} \), the electric field is given by the expression for the field just outside a conductor:

\[
E_{|x| < 5.00 \, \text{m}} = \frac{\sigma}{\varepsilon_0}
\]

Substitute numerical values and evaluate \( E_{|x| < 5.00 \, \text{m}} \):

\[
E_{|x| < 5.00 \, \text{m}} = \frac{1.60 \, \mu \text{C/m}^2}{8.854 \times 10^{-12} \, \text{C/N} \cdot \text{m}^2} = 180.7 \, \text{kN/C} = 181 \, \text{kN/C}
\]

For \( x > 0 \), \( E_{|x| < 5.00 \, \text{m}} \) is in the \(+x\) direction and for \( x < 0 \), \( E_{|x| < 5.00 \, \text{m}} \) is in the \(-x\) direction.
(b) The resultant electric field in Region II is the sum of the fields due to the infinite nonconducting sheet and the charge on the surfaces of the thin square conducting sheet:

\[
\vec{E}_{\text{II}} = \vec{E}_{\text{infinite sheet}} + \vec{E}_{\text{left}} + \vec{E}_{\text{right}},
\]

\[
\frac{\sigma_{\text{infinite sheet}}}{2 \varepsilon_0} \hat{i} - \frac{\sigma_{\text{left}}}{2 \varepsilon_0} \hat{i} - \frac{\sigma_{\text{right}}}{2 \varepsilon_0} \hat{i} = \left( \frac{\sigma_{\text{infinite sheet}} - \sigma_{\text{left}} - \sigma_{\text{right}}}{2 \varepsilon_0} \right) \hat{i}
\]

Due to the presence of the infinite nonconducting sheet, the charges on the thin square conducting sheet are redistributed on the left and right surfaces. The net charge on the thin square conducting sheet is the sum of the charges on its left- and right-hand surfaces:

\[
Q_{\text{left}} + Q_{\text{right}} = 80.0 \mu C
\]

where we’ve assumed that \( Q_{\text{left}} \) and \( Q_{\text{right}} \) are both positive.

Writing this equation in terms of the surface charge densities yields:

\[
\sigma_{\text{left}} + \sigma_{\text{right}} = \frac{Q_{\text{left}} + Q_{\text{right}}}{A} = \frac{80.0 \mu C}{(5.00 \text{ m})^2} = 3.20 \mu C/\text{m}^2
\]

where \( A \) is the area of one side of the thin square conducting sheet.

Because the electric field is zero inside the thin square conducting sheet:

\[
\frac{\sigma_{\text{infinite sheet}}}{2 \varepsilon_0} + \frac{\sigma_{\text{left}}}{2 \varepsilon_0} - \frac{\sigma_{\text{right}}}{2 \varepsilon_0} = 0
\]

or

\[
2.00 \mu C/\text{m}^2 + \sigma_{\text{left}} - \sigma_{\text{right}} = 0
\]

Solving equations (1) and (2) simultaneously yields:

\[
\sigma_{\text{left}} = 0.60 \mu C/\text{m}^2
\]

and

\[
\sigma_{\text{right}} = 2.60 \mu C/\text{m}^2
\]
Substitute numerical values and evaluate $\vec{E}_{\text{II}}$:

$$\vec{E}_{\text{II}} = \left( \frac{2.00 \ \mu \text{C} \ \text{m}^2 - 0.60 \ \mu \text{C} \ \text{m}^2 - 2.60 \ \mu \text{C} \ \text{m}^2}{2 \left( 8.854 \times 10^{-12} \ \text{C}^2 \ \text{N} \cdot \text{m}^2 \right)} \right) \hat{i} = \left( -\frac{67.8 \ \text{kN}}{\text{C}} \right) \hat{i}$$

The resultant electric field in Region IV is the sum of the fields due to the charge on the infinite nonconducting sheet and the charges on the two surfaces of the thin square conducting sheet:

$$\vec{E}_{\text{IV}} = \vec{E}_{\text{infinite sheet}} + \vec{E}_{\text{left}} + \vec{E}_{\text{right}}$$

$$= \frac{\sigma_{\text{infinite sheet}}}{2 \ \varepsilon_0} \hat{i} + \frac{\sigma_{\text{left}}}{2 \ \varepsilon_0} \hat{i} + \frac{\sigma_{\text{right}}}{2 \ \varepsilon_0} \hat{i}$$

$$= \left( \frac{\sigma_{\text{infinite sheet}} + \sigma_{\text{left}} + \sigma_{\text{right}}}{2 \ \varepsilon_0} \right) \hat{i}$$

Substitute numerical values and evaluate $\vec{E}_{\text{IV}}$:

$$\vec{E}_{\text{IV}} = \left( \frac{2.00 \ \mu \text{C} \ \text{m}^2 + 0.60 \ \mu \text{C} \ \text{m}^2 + 2.60 \ \mu \text{C} \ \text{m}^2}{2 \left( 8.854 \times 10^{-12} \ \text{C}^2 \ \text{N} \cdot \text{m}^2 \right)} \right) \hat{i} = \left( \frac{294 \ \text{kN}}{\text{C}} \right) \hat{i}$$

Substitute numerical values and evaluate $\vec{E}_{\text{IV}}$:

$$\vec{E}_{\text{IV}} = \left( \frac{2.00 \ \mu \text{C} \ \text{m}^2 + 0.60 \ \mu \text{C} \ \text{m}^2 + 2.60 \ \mu \text{C} \ \text{m}^2}{2 \left( 8.854 \times 10^{-12} \ \text{C}^2 \ \text{N} \cdot \text{m}^2 \right)} \right) \hat{i} = \left( \frac{294 \ \text{kN}}{\text{C}} \right) \hat{i}$$

**General Problems**

67  A large, flat, nonconducting, non-uniformly charged surface lies in the $x = 0$ plane. At the origin, the surface charge density is $+3.10 \ \mu \text{C}/\text{m}^2$. A small distance away from the surface on the positive $x$ axis, the $x$ component of the electric field is $4.65 \times 10^5 \ \text{N/C}$. What is $E_x$, a small distance away from the surface on the negative $x$ axis?
**Picture the Problem** The electric field just to the right of the large, flat, nonconducting, nonuniformly charged surface is $\sigma/2 \varepsilon_0$ and the electric field just to the left of the surface is $-\sigma/2 \varepsilon_0$. We can express the electric field on both sides of the surface in terms of $E_0$, the electric field in the region in the absence of the charged surface, and then eliminate $E_0$ from these equations to obtain an expression for $E_x$ a small distance away from the surface on the negative $x$ axis.

The electric field on the positive $x$ axis is given by:

$$E_{x>0} = E_0 + \frac{\sigma}{2 \varepsilon_0} \Rightarrow E_0 = E_{x>0} - \frac{\sigma}{2 \varepsilon_0}$$

The electric field on the negative $x$ axis is given by:

$$E_{x<0} = E_0 - \frac{\sigma}{2 \varepsilon_0}$$

Substituting for $E_0$ in the expression for $E_{x<0}$ and simplifying gives:

$$E_{x<0} = E_{x>0} - \frac{\sigma}{2 \varepsilon_0} - \frac{\sigma}{2 \varepsilon_0}$$

$$= E_{x>0} - \frac{\sigma}{\varepsilon_0}$$

Substitute numerical values and evaluate $E_{x,\text{neg}}$:

$$E_{x,\text{neg}} = 4.65 \times 10^4 \text{ N/C} - \frac{3.10 \mu\text{C/m}^2}{8.854 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2} = 115 \text{ kN/C}$$

69 A thin, non-conducting, uniformly charged spherical shell of radius $R$ (Figure 22-44a) has a total positive charge of $Q$. A small circular plug is removed from the surface. (a) What is the magnitude and direction of the electric field at the center of the hole? (b) The plug is now put back in the hole (Figure 22-44b). Using the result of Part (a), find the electric force acting on the plug. (c) Using the magnitude of the force, calculate the "electrostatic pressure" (force/unit area) that tends to expand the sphere.

**Picture the Problem** If the patch is small enough, the field at the center of the patch comes from two contributions. We can view the field in the hole as the sum of the field from a uniform spherical shell of charge $Q$ plus the field due to a small patch with surface charge density equal but opposite to that of the patch cut out.

(a) Express the magnitude of the electric field at the center of the hole:

$$E = E_{\text{spherical shell}} + E_{\text{hole}}$$

Apply Gauss’s law to a spherical gaussian surface just outside the given sphere:

$$E_{\text{spherical shell}} (4\pi r^2) = \frac{Q_{\text{enclosed}}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$
Solve for $E_{\text{spherical shell}}$ to obtain:

$$E_{\text{spherical shell}} = \frac{Q}{4\pi \varepsilon_0 r^2}$$

The electric field due to the small hole (small enough so that we can treat it as a plane surface) is:

$$E_{\text{hole}} = -\frac{\sigma}{2 \varepsilon_0}$$

Substitute for $E_{\text{spherical shell}}$ and $E_{\text{hole}}$ and simplify to obtain:

$$E = \frac{Q}{4\pi \varepsilon_0 r^2} + \frac{-\sigma}{2 \varepsilon_0} = \frac{Q}{4\pi \varepsilon_0 r^2} - \frac{\sigma}{2 \varepsilon_0 \left(\frac{4\pi r^2}{2}\right)}$$

$$= \frac{Q}{8\pi \varepsilon_0 r^2} \text{ radially outward}$$

(b) Express the force on the patch:

$$F = qE$$

where $qE$ is the charge on the patch.

Assuming that the patch has radius $a$, express the proportion between its charge and that of the spherical shell:

$$\frac{q}{\pi a^2} = \frac{Q}{4\pi r^2} \text{ or } q = \frac{a^2}{4r^2} Q$$

Substitute for $q$ and $E$ in the expression for $F$ to obtain:

$$F = \left(\frac{a^2}{4r^2} Q\right) \left(\frac{Q}{8\pi \varepsilon_0 r^2}\right)$$

$$= \frac{Q^2 a^2}{32\pi \varepsilon_0 r^4} \text{ radially outward}$$

(c) The pressure is the force exerted on the patch divided by the area of the patch:

$$P = \frac{32\pi \varepsilon_0 r^4}{\pi a^2} = \frac{Q^2}{32\pi^2 \varepsilon_0 r^4}$$

71 Two identical square parallel metal plates each have an area of 500 cm$^2$. They are separated by 1.50 cm. They are both initially uncharged. Now a charge of +1.50 nC is transferred from the plate on the left to the plate on the right and the charges then establish electrostatic equilibrium. (Neglect edge effects.)

(a) What is the electric field between the plates at a distance of 0.25 cm from the plate on the right? (b) What is the electric field between the plates a distance of 1.00 cm from the plate on the left? (c) What is the electric field just to the left of the plate on the left? (d) What is the electric field just to the right of the plate to the right?
**Picture the Problem** The transfer of charge from the plate on the left to the plate on the right leaves the plates with equal but opposite charges. The symbols for the four surface charge densities are shown in the figure. The $x$ component of the electric field due to the charge on surface 1L is $-\sigma_{1L}/(2\varepsilon_0)$ at points to the left of surface 1L and is $+\sigma_{1L}/(2\varepsilon_0)$ at points to the right of surface 1L, where the $+x$ direction is to the right. Similar expressions describe the electric fields due to the other three surface charges. We can use superposition of electric fields to find the electric field in each of the three regions.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
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</thead>
<tbody>
<tr>
<td>$\sigma_{1L}$</td>
<td>$\sigma_{1R}$</td>
<td>$\sigma_{2L}$</td>
</tr>
</tbody>
</table>

Define $\sigma_1$ and $\sigma_2$ so that:

\[
\sigma_1 = \sigma_{1L} + \sigma_{1R} \quad \text{and} \quad \sigma_2 = \sigma_{2L} + \sigma_{2R}
\]

(a) and (b) In the region between the plates (region II):

\[
E_{x,II} = \frac{\sigma_{1L}}{2\varepsilon_0} + \frac{\sigma_{1R}}{2\varepsilon_0} - \frac{\sigma_{2L}}{2\varepsilon_0} - \frac{\sigma_{2R}}{2\varepsilon_0} = 0 + \frac{\sigma_1}{2\varepsilon_0} - \frac{\sigma_2}{2\varepsilon_0} - 0 = \frac{\sigma_1 - \sigma_2}{\varepsilon_0}
\]

Let $\sigma_2 = -\sigma_1 = \sigma$. Then:

\[
\sigma_1 - \sigma_2 = -\sigma - \sigma = -2\sigma
\]

Substituting for $\sigma_1 - \sigma_2$ and using the definition of $\sigma_2$ yields:

\[
E_{x,II} = -\frac{2\sigma}{\varepsilon_0} = -\frac{Q}{A\varepsilon_0}
\]

Substitute numerical values and evaluate $E_{x,II}$:

\[
E_{x,II} = -\frac{1.50\text{nC}}{8.854\times10^{-12}\text{ C}^2\text{ N}^{-1}\text{ m}^2(500\times10^{-6}\text{ m}^2)} = 339\text{ kN/C}
\]

toward the left
(c) The electric field strength just to the left of the plate on the left (region I) is given by:

\[
E_{x,1} = -\frac{\sigma_{1L}}{2\varepsilon_0} - \frac{\sigma_{1R}}{2\varepsilon_0} - \frac{\sigma_{2L}}{2\varepsilon_0} - \frac{\sigma_{2R}}{2\varepsilon_0}
\]

\[
= 0 - \frac{\sigma_1}{2\varepsilon_0} - \frac{\sigma_2}{2\varepsilon_0} - 0
\]

\[
= -\frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} = 0
\]

(d) The electric field strength just to the right of the plate on the right (region III) is given by:

\[
E_{x,\text{III}} = \frac{\sigma_{1L}}{2\varepsilon_0} + \frac{\sigma_{1R}}{2\varepsilon_0} + \frac{\sigma_{2L}}{2\varepsilon_0} + \frac{\sigma_{2R}}{2\varepsilon_0}
\]

\[
= 0 + \frac{\sigma_1}{2\varepsilon_0} + \frac{\sigma_2}{2\varepsilon_0} - 0
\]

\[
= -\frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = 0
\]

Remarks: If we start with the fact that free charges are only found on the surfaces of the plates facing each other, then a much simpler solution is possible. Any plane of charge produces a field \(\sigma/2\varepsilon_0\) perpendicular to the plane. The field in region III directed everywhere away from the plane and the field of the left plane is everywhere directed toward it.

73 A quantum-mechanical treatment of the hydrogen atom shows that the electron in the atom can be treated as a smeared-out distribution of negative charge of the form \(\rho(r) = -\rho_0 e^{-2r/a}\). Here \(r\) represents the distance from the center of the nucleus and \(a\) represents the first Bohr radius which has a numerical value of 0.0529 nm. Recall that the nucleus of a hydrogen atom consists of just one proton and treat this proton as a positive point charge. (a) Calculate \(\rho_0\), using the fact that the atom is neutral. (b) Calculate the electric field at any distance \(r\) from the nucleus.

**Picture the Problem** Because the atom is uncharged, we know that the integral of the electron’s charge distribution over all of space must equal its charge \(q_e\). Evaluation of this integral will lead to an expression for \(\rho_0\). In (b) we can express the resultant electric field at any point as the sum of the electric fields due to the proton and the electron cloud.

(a) Because the atom is uncharged, the integral of the electron’s charge distribution over all of space must equal its charge \(e\):

\[
e = \int_0^\infty \rho(r)dV = \int_0^\infty \rho(r)4\pi r^2dr
\]
Substitute for $\rho(r)$ and simplify to obtain:

$$e = -\int_{0}^{\infty} \rho_0 e^{-2y/a} 4\pi r^2 dr$$

$$= -4\pi \rho_0 \int_{0}^{\infty} r^2 e^{-2y/a} dr$$

Use integral tables or integration by parts to obtain:

$$\int_{0}^{\infty} r^2 e^{-2y/a} dr = \frac{a^3}{4}$$

Substitute for $\int_{0}^{\infty} r^2 e^{-2y/a} dr$ to obtain:

$$e = -4\pi \rho_0 \left( \frac{a^3}{4} \right) = -\pi a^3 \rho_0$$

Solving for $\rho_0$ yields:

$$\rho_0 = \frac{e}{\pi a^3}$$

(b) The field will be the sum of the field due to the proton and that of the electron charge cloud:

$$E = E_p + E_{\text{cloud}}$$

Express the field due to the electron cloud:

$$E_{\text{cloud}}(r) = \frac{kQ(r)}{r^2}$$

where $Q(r)$ is the net negative charge enclosed a distance $r$ from the proton.

Substitute for $E_p$ and $E_{\text{cloud}}$ to obtain:

$$E(r) = \frac{ke}{r^3} + \frac{kQ(r)}{r^2}$$

(1)

$Q(r)$ is given by:

$$Q(r) = \int_{0}^{r} 4\pi r'^2 \rho(r')dr'$$

$$= 4\pi \int_{0}^{r} r'^2 \rho_0 e^{-2y/a} dr'$$

From Part (a), $\rho_0 = \frac{e}{\pi a^3}$:

$$Q(r) = 4\pi \left( \frac{e}{\pi a^3} \right) \int_{0}^{r} r'^2 e^{-2y/a} dr'$$

$$= \frac{4e}{a^3} \int_{0}^{r} r'^2 e^{-2y/a} dr'$$
From a table of integrals:

\[
\int_{0}^{r} x^2 e^{-2x/a} \, dx = \frac{1}{4} e^{-2r/a} a \left[ e^{-2r/a} - 1 \right] a^2 - 2ar - 2r^2 \\
= \frac{1}{4} e^{-2r/a} a^3 \left( e^{-2r/a} - 1 \right) - 2 \frac{r}{a} - \frac{r^2}{a^2} \\
= \frac{a^3}{4} \left[ 1 - e^{-2r/a} \right] - 2e^{-2r/a} \left( \frac{r}{a} + \frac{r^2}{a^2} \right)
\]

Substituting for \( \int_{0}^{r} r^2 e^{-2r/a} \, dr \) in the expression for \( Q(r) \) and simplifying yields:

\[
Q(r) = -\frac{e}{4} \left[ 1 - e^{-2r/a} \right] - 2e^{-2r/a} \left( \frac{r}{a} + \frac{r^2}{a^2} \right)
\]

Substitute for \( Q(r) \) in equation (1) and simplify to obtain:

\[
E(r) = \frac{ke}{r^2} - \frac{ke}{4r^2} \left[ 1 - e^{-2r/a} \right] - 2e^{-2r/a} \left( \frac{r}{a} + \frac{r^2}{a^2} \right)
\]

\[
= \frac{ke}{r^2} \left[ 1 - \frac{1}{4} \left( 1 - e^{-2r/a} \right) - 2e^{-2r/a} \left( \frac{r}{a} + \frac{r^2}{a^2} \right) \right]
\]

79 •• A uniformly charged, infinitely long line of negative charge has a linear charge density of \(-\lambda\) and is located on the \( z \) axis. A small positively charged particle that has a mass \( m \) and a charge \( q \) is in a circular orbit of radius \( R \) in the \( xy \) plane centered on the line of charge. (a) Derive an expression for the speed of the particle. (b) Obtain an expression for the period of the particle’s orbit.

**Picture the Problem** (a) We can apply Newton’s second law to the particle to express its speed as a function of its mass \( m \), charge \( q \), and the radius of its path \( R \), and the strength of the electric field due to the infinite line charge \( E \).

(b) The period of the particle’s motion is the ratio of the circumference of the circle in which it travels divided by its orbital speed.
(a) Apply Newton’s second law to the particle to obtain:

\[ \sum F_{\text{radial}} = qE = \frac{mv^2}{R} \]

where the inward direction is positive.

Solving for \( v \) yields:

\[ v = \sqrt{\frac{qRE}{m}} \]

The strength of the electric field at a distance \( R \) from the infinite line charge is given by:

\[ E = \frac{2k\lambda}{R} \]

Substitute for \( E \) and simplify to obtain:

\[ v = \sqrt{\frac{2kq\lambda}{m}} \]

(b) The speed of the particle is equal to the circumference of its orbit divided by its period:

\[ v = \frac{2\pi R}{T} \Rightarrow T = \frac{2\pi R}{v} \]

Substitute for \( v \) and simplify to obtain:

\[ T = \pi R \sqrt{\frac{2m}{kq\lambda}} \]

---

81 The charges \( Q \) and \( q \) of Problem 80 are +5.00 \( \mu \)C and −5.00 \( \mu \)C, respectively, and the radius of the ring is 8.00 cm. When the particle is given a small displacement in the \( x \) direction, it oscillates about its equilibrium position at a frequency of 3.34 Hz. (a) What is the particle’s mass? (b) What is the frequency if the radius of the ring is doubled to 16.0 cm and all other parameters remain unchanged?

**Picture the Problem**

Starting with the equation for the electric field on the axis of a ring charge, we can factor the denominator of the expression to show that, for \( x \ll a \), \( E_x \) is proportional to \( x \). We can use \( F_x = qE_x \) to express the force acting on the particle and apply Newton’s second law to show that, for small displacements from equilibrium, the particle will execute simple harmonic motion. Finally, we can find the angular frequency of the motion from the differential equation and use this expression to find the frequency of the motion when the radius of the ring is doubled and all other parameters remain unchanged.

(a) Express the electric field on the axis of the ring of charge:

\[ E_x = \frac{kQx}{(x^2 + a^2)^{3/2}} \]
Factor $a^2$ from the denominator of $E_x$ to obtain:

$$E_x = \frac{kQx}{a^2 \left(1 + \frac{x^2}{a^2}\right)^{3/2}}$$

$$= \frac{kQx}{a^3 \left(1 + \frac{x^2}{a^2}\right)} \approx \frac{kQ}{a^3} x$$

provided $x << a$.

Express the force acting on the particle as a function of its charge and the electric field:

$$F_x = -qE_x = \frac{-kqQ}{a^3} x$$

Because the negatively charged particle experiences a linear restoring force, we know that its motion will be simple harmonic. Apply Newton’s second law to the negatively charged particle to obtain:

$$m \frac{d^2x}{dt^2} = -\frac{kqQ}{a^3} x$$

or

$$\frac{d^2x}{dt^2} + \frac{kqQ}{ma^3} x = 0$$

the differential equation of simple harmonic motion.

The angular frequency of the simple harmonic motion of the particle is given by:

$$\omega = \sqrt{\frac{kqQ}{ma^3}}$$

(1)

Solving for $m$ yields:

$$m = \frac{kqQ}{\omega^2 a^3} = \frac{kqQ}{4\pi^2 f^2 a^3}$$

Substitute numerical values and evaluate $m$:

$$m = \left(8.988 \times 10^9 \frac{N \cdot m^2}{C^2}\right) \left| -5.00 \mu C \right| \left(5.00 \mu C \right)$$

$$= 4\pi^2 \left(3.34 \text{ s}^{-1}\right)^2 \left(8.00 \text{ cm}\right)^3 = 0.997 \text{ kg}$$

(b) Express the angular frequency of the motion if the radius of the ring is doubled:

$$\omega' = \sqrt{\frac{kqQ}{m(2a)^3}}$$

(2)
Divide equation (2) by equation (1) to obtain:

\[
\frac{\omega'}{\omega} = \frac{2\pi f'}{2\pi f} = \frac{\sqrt{\frac{kqQ}{ma^2}}}{\sqrt{\frac{kqQ}{ma^2}}} = \frac{1}{\sqrt{8}}
\]

Solve for \(f'\) to obtain:

\[
f' = f = \frac{3.34 \text{ Hz}}{\sqrt{8}} = \frac{1.18 \text{ Hz}}{\sqrt{8}}
\]

---

Consider a simple but surprisingly accurate model for the hydrogen molecule: two positive point charges, each having charge \(+e\), are placed inside a uniformly charged sphere of radius \(R\), which has a charge equal to \(-2e\). The two point charges are placed symmetrically, equidistant from the center of the sphere (Figure 22-48). Find the distance from the center, \(a\), where the net force on either point charge is zero.

**Picture the Problem** We can find the distance from the center where the net force on either charge is zero by setting the sum of the forces acting on either point charge equal to zero. Each point charge experiences two forces; one a Coulomb force of repulsion due to the other point charge, and the second due to that fraction of the sphere’s charge that is between the point charge and the center of the sphere that creates an electric field at the location of the point charge.

Apply \(\sum F = 0\) to either of the point charges:

\[
F_{\text{Coulomb}} - F_{\text{field}} = 0 \quad (1)
\]

Express the Coulomb force on the proton:

\[
F_{\text{Coulomb}} = \frac{ke^2}{(2a)^2} = \frac{ke^2}{4a^2}
\]

The force exerted by the field \(E\) is:

\[
F_{\text{field}} = eE
\]

Apply Gauss’s law to a spherical surface of radius \(a\) centered at the origin:

\[
E(4\pi a^2) = \frac{Q_{\text{enclosed}}}{\varepsilon_0}
\]

Relate the charge density of the electron sphere to \(Q_{\text{enclosed}}\):

\[
\frac{2e}{\frac{4}{3}\pi R^3} = \frac{Q_{\text{enclosed}}}{\frac{4}{3}\pi a^3} \Rightarrow Q_{\text{enclosed}} = \frac{2ea^3}{R^3}
\]

Substitute for \(Q_{\text{enclosed}}\):

\[
E(4\pi a^2) = \frac{2ea^3}{\varepsilon_0 R^3}
\]

Solve for \(E\) to obtain:

\[
E = \frac{ea}{2\pi \varepsilon_0 R^3} \Rightarrow F_{\text{field}} = \frac{e^2 a}{2\pi \varepsilon_0 R^3}
\]
Substitute for $F_{\text{Coulomb}}$ and $F_{\text{field}}$ in equation (1):

\[
\frac{ke^2}{4a^2} - \frac{e^2a}{2\pi \epsilon_0 R^3} = 0
\]

or

\[
\frac{ke^2}{4a^2} - \frac{2ke^2a}{R^3} = 0 \Rightarrow a = \frac{1}{8} \sqrt{\frac{1}{R}} = \frac{1}{2} R
\]