Take-home packet from PPMTC Academy

prepared by Brian Hopkins, Saint Peter’s College, bhopkins@spc.edu

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1. Take Away (Tuesday p.m., p2 of this packet): game with a variant that was used on MTV.

2. Trains (Tuesday p.m., p3): one and a half chapters from a related book project. The preface includes motivation for exploration and this style of “combinatorial proof.”

3. Figurate Numbers (Wednesday a.m., p25): explorations of and relations between triangular, square, pentagonal numbers, etc.

4. Restricted Trains (Wednesday a.m., p26): notes that continue in the vein of Session 2, far from polished book-style.

5. Balancing (Wednesday p.m., p31): notes on the two puzzles about weighing integer weight objects.


7. Daisy & Kayles (Thursday a.m., p34): variants of Take Away from Session 1 where adjacency matters.

8. Cuisenarea (Thursday a.m., p36): open ended explorations about the number of ways to use Cuisenaire rods to fill rectangular areas.


11. Train Connections (Friday a.m., p51): continued from Session 4.

12. Take Away Revisited (Friday a.m., p52): notes on another variant of the Take Away game of Session 1 with other tie-ins.

I encourage you to adapt anything here for your own use. Please be in touch for any additional information, files of individual pages, etc. Thanks again for your time and work during the Academy.
Session 1: Take 1 or 2

- Setup: 9 stones
- Legal moves: in each turn, take 1 or 2 stones
- Winner: takes the last stone

The second player has a winning strategy here. His goal is to always leave a multiple of 3 stones. That is, if the first player takes 1, he takes 2; if she takes 2, he takes 1.

As students work on this, they may get stuck on even and odd. A common intermediate step is “getting down to” 3 or 4. In such cases, you may suggest that students start with 6 stones instead of 9. A big hint is to group the stones in clusters of 3.

Variations:

- Start with 13 stones instead of 9. (Now first player can win by taking 1, which reduces to a multiples of 3.)
- Arbitrary number of stones. (As appropriate, arithmetic modulo 3 can be introduced here.)
- The mise`re version with 13 stones: whoever takes the last stone loses. (This is equivalent to the standard version of the 12 stone game; if you win that, then your opponent must then take the final “carcinogenic” stone.)
- The flag game from Survivor Thailand: 21 flags, each team takes 1, 2, or 3 in each turn. (The key is now multiples of 4, since taking 1, 2, or 3 can be responded to by taking 3, 2, or 1, respectively. In this game, the first team can win by taking 1 flag and then “responding” to each of the second team’s moves. One can find the Survivor footage online; there is no evidence of strategic play until there are around 6 flags left.)
- Allow each player to take 2 or 3 stones in each turn. There are two ways to modify winning: since neither player may not be able to take the last stone, you might introduce “draw” which is better than losing and worse than winning. A more standard approach is to give the win to whoever makes the last legal move (whether that leaves 0 or 1 stones). With either convention, this game is more involved.

These are restricted versions of a combinatorial game Nim; several other variations are examined later in this packet.
Session 2: Trains—Pascal via Cuisenaire

e_excerpt from a book project, provisional title *Hands-On Combinatorics*

**Preface:** Pascal’s triangle and Fibonacci numbers are among the most well-known and well-beloved objects in mathematics. This book shows how they can be understood in concrete visual ways. We use trains consisting of various length cars to represent these combinatorial objects and to prove things about them in ways that often lead to deeper understanding than purely symbolic proofs. These tools are based on Cuisenaire rods familiar to many elementary school students; some of the results are common in discrete mathematics courses. The combination offers a wealth of combinatorial insight to a wide variety of readers, from high school students excited about new aspects of the subject to mathematicians interested in bijective proofs.

These same tools provide insight into less familiar structures, such as Padovan numbers, “tribonacci” numbers, Pell numbers, Delannoy’s triangle, Jacobsthal numbers, and many more known primarily by their ID on the marvelous resource, the Online Encyclopedia of Integer Sequences. The interconnections between these objects are very rich. At some level this is a book about compositions of integers, with a few references to partitions and “colored compositions.” We also make connections to the “postage stamp problem” of Frobenius, run-length limited codes from symbolic dynamics, a problem from the Putnam exam, and open questions.

Beyond the combinatorial content, though, this book has a more subtle and more important goal: helping the reader understand the *process* that leads to mathematical conjectures and proofs. Reading research articles and sometimes even textbooks can leave you feeling on the outside looking in. Perhaps you can understand the theorem statements and, with work, even follow the proofs, but where did the statements and proofs come from? As J. Alfred Prufrock says in T. S. Eliot’s poem,

*I have heard the mermaids singing, each to each.*

*I do not think that they will sing to me.*

In this book, a mathematician sings to you. Each theorem includes not only a formal statement and (at least one) proof, but also the data that suggests the theorem statement and an exploration of the correspondence that leads to the proof. These additional steps of examining data for patterns and exploring possible explanations are what mathematicians often do along the way to a theorem and proof, but that initial work is seldom shared. My hope is that helping you experience the full process leading to a proof will enable you to sing your own mathematical song.
1 All Trains

1.1 Cars

This book discusses many sequences of numbers that are among the foundations of combinatorics. The tools we will use are trains made up of cars of varying lengths. The shading scheme for the cars is based on Georges Cuisenaire’s rods.

<table>
<thead>
<tr>
<th>length</th>
<th>car</th>
<th>color</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>white</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>dark gray</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>light gray</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>vertical stripe</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>horizontal stripe</td>
</tr>
</tbody>
</table>

While actual Cuisenaire rods stop at length 10, we will assume that there are cars of every positive integer length. A train is one or more cars arranged end to end in a line. Ideally, you will have a set of Cuisenaire rods handy while you go through this book. Drawing boxes on graph paper would work, and the book’s illustrations could suffice, but you’ll do best if you are actively engaged in this representation of the mathematics.

1.2 Counting Trains

How many trains of length 4 are there? Build them and see! We’ll soon determine the number of trains of length \( n \), but the proof will come from noticing patterns in small cases. An initial question is what to do with trains consisting of the same cars in different order. I.e., should all be included? Yes: this is a context “where order matters.” With that clarification, you should be able to come up with the following list.
There are eight trains of length 4. To have more data to work with, consider all trains of length 3.

So there are four trains of length 3. To complete the data going down, there are two trains of length 2 and one train of length 1. That’s a tempting pattern. One suspects that there are sixteen trains of length 5, thirty-two of length 6, etc. Writing $T(n)$ for the number of trains of length $n$, it looks like $T(n) = 2^{n-1}$. But there are many situations where tempting patterns do not hold. We want to explain why this pattern holds, prove that it is correct.

**Exploration.** Let’s focus on the transition from trains of length 3 to trains of length 4. There are twice as many trains of length 4, so we would like to take two sets of all length 3 trains and somehow make all the length 4 trains.

One way to make a length 4 train from a length 3 train is to add a white car on the end:

What’s left? We need to somehow associate our second set of length 3 trains with the remaining length four trains:
How can we describe this operation? Some of the cars changed. In particular, in the first train, the light gray car (length 3) became one with vertical stripes (length 4). In the second train, the first car stayed the same and the white car (length 1) became dark gray (length 2). In the third train, the first car stayed the same and the dark gray car (length 2) became light gray (length 3). Finally, in the fourth train, the first two cars stayed the same and the last white car became dark gray. In general, the last car was extended one unit of length, which makes it change to the “next color.”

These two operations, adding a white and extending the last car, are the ingredients for showing the general result about the number of trains of length $n$. There are some more details we will need for a solid proof.

**Theorem 1.** $T(n) = 2^{n-1}$

**Proof 1:** This is a proof by induction. Since there is one train of length 1, and $2^0 = 1$, the claim is true for $n = 1$. This is the base case.

The induction hypothesis assumes that the claim is true for $n - 1$, i.e., that $T(n - 1) = 2^{n-2}$. We will show that $T(n) = T(n - 1) + T(n - 1)$. This will finish the proof, since then we will have $T(n) = 2 \cdot T(n - 1) = 2 \cdot 2^{n-2} = 2^{n-1}$.

Given two sets of all length $n - 1$ trains, we want to build all length $n$ trains. On the first set of length $n - 1$ trains, add a white car on the right hand side. On the second set of length $n - 1$ trains, extend the rightmost car to be one longer. Both procedures produce length $n$ trains. Could there be any overlap? That is, could the same train come up under both operations? No: every train produced from the first set has a white car at the end, while every train from the second set has a dark gray or longer car at the end. (This shows that the correspondence between two sets of length $n - 1$ trains and the length $n$ trains is “one to one” or an “injection.”)

We have shown that there are at least $T(n - 1) + T(n - 1)$ trains of length $n$. Could there be more length $n$ trains that do not arise from the correspondence? No, every length $n$ train is built by one of the two operations: For a length $n$ train $U$ that ends in a white car, removing that white cars leaves a length $n - 1$ train whose image under the correspondence is $U$. For a length $n$ train $V$ that ends in a dark gray or longer car, decreasing that car to be one shorter leaves a length $n - 1$ train whose image under the correspondence is $V$. (This shows that the correspondence is “onto” or a “surjection.”) A map that is both an injection and a surjection is called a “bijection.”)

We conclude that the two operations provide an exact correspondence between two sets of length $n - 1$ trains and the length $n$ trains. As detailed before, this concludes the proof. $\square$
The careful reader will have noticed the designation “Proof 1.” Why have more than one proof? At this point we’re completely sure that the result is true. Another proof can highlight another way of looking at the situation and provide deeper understanding. That is, we can be interested in a proof for more than establishing the truth of a claim.

**Exploration.** The formula \( T(n) = 2^{n-1} \) begs the question “what are there \( n - 1 \) of?” The formula would make sense if a length \( n \) train were associated with \( n - 1 \) two-fold decisions.

Consider building a length 3 train directly. There are two internal junctures (shown with thin lines) where a decision must be made to make a cut between cars or join the pieces on either side to make a larger car. Below are the correspondences between the \( 2^2 \) possible ordered pairs of decisions and length 3 trains.

```
join, join
join, cut
cut, join
cut, cut
```

This is the insight needed for our second proof of the \( T(n) = 2^{n-1} \) theorem.

**Proof 2:** A length \( n \) train has \( n - 1 \) internal junctures. Moving left to right, making \( n - 1 \) decisions between cut and join for each juncture determines a length \( n \) train. Clearly, two different sequences of decisions give rise to different trains, and every train corresponds to a sequence. Since there are two possible decisions at each juncture, there are \( 2^{n-1} \) different sequences of these binary decisions, thus \( 2^{n-1} \) trains of length \( n \).

The cut / join sequences could be thought of as a series of coin tosses or a binary string. In fact, most of the topics covered in this book could be done in the context of sequences of 0’s and 1’s, but the colored rods are more engaging.

Before moving on, what is the connection between the two proofs? In the induction of Proof 1, we could imagine the train of length \( n - 1 \) being given by a list of \( n - 2 \) cut or join choices. The operation of adding a white car corresponds to choosing cut for the \((n - 1)\)st juncture. The operation of extending the rightmost car by one corresponds to choosing join instead.

### 1.3 A Putnam Problem

The William Lowell Putnam Mathematics Competition is a challenging examination offered every December to North American college students. Problem A2 from the 2005 test can be solved by using what we already know about trains.
Let $S = \{(a, b) : a = 1, 2, ..., n; b = 1, 2, 3\}$. A rook tour of $S$ is a polygonal path made up of line segments connecting points $p_1$, $p_2$, ..., $p_{3n}$ in sequence such that

(i) $p_i \in S$,

(ii) $p_i$ and $p_{i+1}$ are a unit distance apart, for $1 \leq i < 3n$,

(iii) for each $p \in S$ there is a unique $i$ such that $p_i = p$.

How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$? (An example of such a rook tour for $n = 5$ is depicted.)
2 Pascal’s Triangle

2.1 Counting Trains by Cars

We will see throughout this book that various groupings of trains give rise to important integer sequences and structures. The first of these comes from grouping the trains of length $n$ by the number of cars $k$ in the trains. In the figure, the leftmost column corresponds to $k = 1$, trains consisting of a single car, then the $k = 2$ column for trains consisting of two cars, etc. The rows correspond to trains of length $n$ for $n = 1, \ldots, 5$.

Counting the number of trains in each position gives a familiar structure:

\[
\begin{array}{cccccc}
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

Pascal’s triangle. (We will usually present triangles of numbers in right triangles as shown, not the more common equilateral triangles.) There’s a theorem here. Let $T(n, k)$ be the number of length $n$ trains consisting of $k$ cars.

We assume that you are familiar with the binomial coefficients, also known as “choose numbers,” that constitute Pascal’s triangle. For example, $\binom{4}{2}$ counts the number of ways to choose two out of four objects where the order of selection does not matter. It evaluates as 6, found in the bottom row of the table above. Remember the convention that $\binom{n}{0} = 1$ for all $n \geq 0$; there is one way to choose nothing. The portion of Pascal’s Triangle shown above corresponds to binomial coefficients as follows.
Exploration. How does the binomial coefficient \( \binom{4}{2} = 6 \) correspond to the six length 5 trains with 3 cars in the illustration? Consider the sequence of cut and join decisions corresponding to each train.

These are all the sequences of 4 cuts and joins that include exactly 2 cuts. Four decisions give trains of length 5. Two of the decisions being cut gives trains consisting of 3 cars. Now we can state and prove the theorem establishing the general connection between trains and Pascal’s triangle.

**Theorem 2.** \( T(n, k) = \binom{n-1}{k-1} \)

**Proof:** As explained in Proof 2 of Theorem 1, a train of length \( n \) corresponds to a sequence of \( n - 1 \) decisions between cut and join. A train with \( k \) cars has exactly \( k - 1 \) cuts in that sequence. Building a length \( n \) train with \( k \) cars, then, is equivalent to choosing which \( k - 1 \) of the \( n - 1 \) ordered decisions will be cut (the rest are all join). There are \( \binom{n-1}{k-1} \) ways to do this, thus that many length \( n \) trains consisting of \( k \) cars. \( \square \)

This correspondence between certain trains and binomial coefficients is one of the key ingredients for the combinatorial proofs that fill this book. You have probably explored patterns in Pascal’s triangle, and perhaps even proved them algebraically. In the next section, we establish many patterns in Pascal’s triangle by using operations on trains.

### 2.2 Pascal’s Formula and Initial Identities

The fundamental structure of Pascal’s triangle is that an entry is the sum of two numbers in the row above it (allowing for implicit zeroes on the edges). This
is often how Pascal’s triangle is defined. However, since we started with trains, we need to prove that this relation holds. Again, a combinatorial proof in terms of trains helps explain why the formula is true.

**Theorem 3, Pascal’s Formula.** \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \).
In terms of trains, this says \( T(n + 1, k + 1) = T(n, k) + T(n, k + 1) \).

**Exploration.** Before proving the formula, consider the ten length 6 trains consisting of 3 cars. We want to relate them to the four length 5 trains consisting of 2 cars and the six length 5 trains consisting of 3 cars.

Notice that four of the ten length 6 trains containing 3 cars end in a white car, and there are four length 5 trains containing 2 cars. The six remaining trains of each side have three cars, differing exactly by one in the length of the last car. We have the same relations used in Theorem 1: add a white car to the length 5 trains that have too few cars, extend by one the last car of the length 5 trains that have the right number of cars. Here is the correspondence, row by row.

The proof of Pascal’s Formula is much like the first proof of Theorem 1.

**Proof of Pascal’s Formula:** We show \( T(n+1, k+1) = T(n, k) + T(n, k+1) \) by building a bijection between the corresponding trains.

For each length \( n \) train consisting of \( k \) cars, add a white length 1 car to the right-hand side to produce a length \( n + 1 \) train consisting of \( k + 1 \) cars. For each length \( n \) train consisting of \( k + 1 \) cars, extend the rightmost car by one to produce a length \( n + 1 \) train still consisting of \( k + 1 \) cars.
Notice that each of the first set of length \(n+1\) trains end in a white car while each of the second set of length \(n+1\) trains end in a dark gray or longer car, so the two sets have no overlap. Since each set of length \(n\) trains we started with consist of distinct trains, the resulting length \(n+1\) trains are distinct.

Going the other direction, could there be a length \(n+1\) train consisting of \(k+1\) cars that cannot be constructed as described above? No: every length \(n+1\) train with \(k+1\) cars either ends in a white car, so that its removal leaves a length \(n\) train with \(k\) cars, or end in a dark gray or longer car, so that reducing it leaves a length \(n\) train with \(k+1\) cars.

Note that the proof is still valid for \(k = 0\) or \(k = n\), when one of the summands is zero. The length \(n+1\) train consisting of 1 car is built by extending the length \(n\) train consisting of 1 car, and not by adding any white cars. The length \(n+1\) train consisting of \(n+1\) cars is built by adding a white car to the length \(n\) train consisting of \(n\) cars, and not by extending any cars. \(\square\)

There are many immediate corollaries of Pascal’s Formula; here is one with an imaginative name derived from the shape of the relevant terms in Pascal’s Triangle.

\[
\begin{array}{ccccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]

**Theorem 4, Hockey Stick.** \(\binom{n-k-1}{0} + \binom{n-k}{1} + \cdots + \binom{n-1}{k} = \binom{n}{k}\) for \(k < n\).

In terms of trains, \(T(n-k,1)+T(n-k+1,2)+\cdots+T(n,k+1) = T(n+1,k+1)\).

**Exploration.** Let’s look at a smaller example than the numbers indicated above. The \(n = 5\), \(k = 2\) case of the theorem gives \(1 + 3 + 6 = 10\), in terms of trains, \(T(3,1) + T(4,2) + T(5,3) = T(6,3)\). This is a somewhat familiar illustration.
As in the example of Pascal’s Formula, the length 5 trains consisting of 3 cars already have the correct number of cars and extending the last car matches \( T(5, 3) = 6 \) of the length 6 trains with 3 cars. This leaves the following trains to be associated.

Notice that three of the trains on the right hand side end with a single white car, and one with two white cars. For the length 4 trains, they have analogs in the right hand side with the same first car, a second car extended by one, and an extra white car. The length 3 train corresponds then to the length 6 train that begins with a vertical striped car and is followed by two whites.

Generalizing, it looks like the pattern is to extend the last car of the shorter trains by one and add whites to increase both the length and number of cars. For the reverse map, moving right to left, remove any white cars and decrease the length of the first dark gray or longer car by 1. As an example, consider how \( T(6, 4) = 10 \), grouped by terminal white cars, corresponds to \( T(5, 4) + T(4, 3) + T(3, 2) + T(2, 1) = 4 + 3 + 2 + 1 \).

Write \( T(n, k) \) for the length \( n \) trains having \( k \) cars, so that \( T(n, k) \) is a set having \( T(n, k) \) elements.

**Proof of the Hockey Stick Theorem**: We establish a bijection between the set of trains \( H = \{ T(n - k, 1), T(n - k + 1, 2), \ldots, T(n, k + 1) \} \) and \( T(n + 1, k + 1) \), the length \( n + 1 \) trains having \( k + 1 \) cars. Notice that the trains in \( H \) are exactly the ones satisfying \( \text{length} - \text{number of cars} = n - k - 1 \).

Let \( U \in H \) have \( \ell \) cars (note \( 1 \leq \ell \leq k+1 \)), so that its length is \( n - k - 1 + \ell \). Define \( \varphi : H \to T(n + 1, k + 1) \) as follows: extend the rightmost car of \( \varphi(U) \) by one and then add \( (k + 1 - \ell) \) length 1 white cars on the right-hand side (this could be 0). This \( \varphi(U) \) has \( \ell + (k + 1 - \ell) = k + 1 \) cars and length

\[
\underbrace{(n - k - 1 + \ell)}_{\text{length of } U} + \underbrace{1}_{\text{extended car}} + \underbrace{(k + 1 - \ell)}_{\text{additional white cars}} = n + 1
\]

establishing \( \varphi(U) \in T(n + 1, k + 1) \).
Now define \( \chi : T(n+1, k+1) \to H \). Given \( V \in T(n+1, k+1) \), let \( m \) be the number of adjacent white cars on the right-hand side. Since \( k+1 \leq n \), there must be at least one car dark gray or longer, so that \( 0 \leq m \leq k \). Remove these \( m \) white cars from the right-hand side of \( V \), and then decrease the rightmost remaining car by one. The resulting \( \chi(V) \) has length \( (n+1) - m - 1 = n - m \) and \( (k+1) - m \). Since \( (n-m) - (k+1-m) = n-k-1 \), we have \( \chi(V) \in H \).

From the definitions, performing \( \varphi \) and then \( \chi \) is the identity on \( H \), and \( \chi \) followed by \( \varphi \) is the identity on \( T(n+1, k+1) \). This established a bijection between the two sets of trains.

One very apparent structure of Pascal’s triangle is that the rows are “palindromes,” i.e., they read the same left to right as right to left.

**Theorem 5, Row Symmetry.** \( \binom{n}{k} = \binom{n}{n-k} \).

In terms of trains, \( T(n+1, k+1) = T(n+1, n+1-k) \).

**Exploration.** For example, \( T(5,1) = T(5,5) \) and \( T(5,2) = T(5,4) \). Since the number of cars is determined by the number of cuts, consider the corresponding cut / join decisions.

Reordering the elements of \( T(5,4) \) shows the correspondence: each cut / join decision is reversed. We will use this “conjugation” operation many times, so we give it a symbol. Let \( U \in T(n+1) \) be given by a sequence of cut / join decisions \( \{u_1, \ldots, u_n\} \). Define the over-line operation as \( \overline{\text{cut}} = \text{join} \) and \( \overline{\text{join}} = \text{cut} \). Let \( \overline{U} \in T(n+1) \) be the train corresponding to the sequence \( \{\overline{u_1}, \ldots, \overline{u_n}\} \). Each row above is an example of two trains that are conjugates of each other.

**Proof of the Row Symmetry Theorem:** We give a bijection between \( T(n+1, k+1) \) and \( T(n+1, n+1-k) \). Given \( U \in T(n+1, k+1) \), its cut / join sequence consists of \( k \) cuts and \( n-k \) joins in some order. Therefore, its conjugate \( \overline{U} \) is determined by a sequence with \( n-k \) cuts and \( k \) joins, so \( \overline{U} \in T(n+1, n+1-k) \). Applying conjugation twice is equivalent to the identity map, so this establishes the bijection.

What does conjugation do to the trains in \( T(5,3) \)? The conjugate of a length 5 train with 3 cars will be a length 5 train with 3 cars, but since each cut / join decision is changed, it cannot be the same train. Here is the correspondence.
The set $T(5,3)$ consists of conjugate pairs. This suggests a parity result for certain numbers in Pascal’s triangle.

**Corollary.** $\binom{2n}{n}$ is an even number.

In terms of trains, $T(2n+1, n+1)$ is an even number.

**Proof:** Given a train $U \in T(2n+1, n+1)$, we know $U$ also has length $2n+1$ with $(2n+1) - n = n+1$ cars but $\overline{U} \neq U$. Therefore, conjugation partitions $T(2n+1, n+1)$ into pairs, i.e., $T(2n+1, n+1)$ is an even number.

The first few examples are $\binom{2}{1} = 2$, $\binom{4}{2} = 6$, $\binom{6}{3} = 20$, $\binom{8}{4} = 70$, and $\binom{10}{5} = 252$. These are called “central binomial coefficients” and will come up again in the next section.

### 2.3 Row Sum Identities

There is a nice pattern in the sum of each row in Pascal’s triangle.

\[
\begin{align*}
1 &= 2^0 \\
1 + 1 &= 2 = 2^1 \\
1 + 2 + 1 &= 4 = 2^2 \\
1 + 3 + 3 + 1 &= 8 = 2^3 \\
1 + 4 + 6 + 4 + 1 &= 16 = 2^4
\end{align*}
\]

The way we have built Pascal’s triangle in terms of trains, this result comes “for free.”

**Theorem 6, Row Sums.** $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

In terms of trains, $T(n+1, 1) + T(n+1, 2) + \cdots + T(n+1, n+1) = 2^n$.

**Proof:** The sum $T(n+1, 1) + \cdots + T(n+1, n+1)$ represents all length $n+1$ trains grouped by number of cars. Thus $T(n+1, 1) + \cdots + T(n+1, n+1) = T(n+1)$ and by Theorem 1 we know $T(n+1) = 2^n$.

We will explore three variations of row sums to see how trains can explain other patterns of Pascal’s triangle.

#### 2.3.1 Alternating Row Sums

Instead of summing the entries of each row, consider what happens when we alternate between adding and subtracting (starting with the second row).
\[
\begin{align*}
1 - 1 &= 0 \\
1 - 2 + 1 &= 0 \\
1 - 3 + 3 - 1 &= 0 \\
1 - 4 + 6 - 4 + 1 &= 0
\end{align*}
\]

For some rows, it’s clear why this happens: the same numbers are being added and subtracted. The result is not obvious for the other rows, though, such as \(1 - 4 + 6 - 4 + 1 = 0\).

**Exploration.** In terms of trains, this equation is \(T(5, 1) - T(5, 2) + T(5, 3) - T(5, 4) + T(5, 5) = 0\). For a combinatorial proof, we move the subtracted terms to the right-hand side: \(T(5, 1) + T(5, 3) + T(5, 5) = T(5, 2) + T(5, 4)\). We want a bijection between the length 5 trains with an odd number of cars and the length 5 trains with an even number of cars.

![Diagram](image)

The right-hand column of length 5 trains with an even number of cars has been reordered to show the following pattern: on each row, the two trains differ only by whether the rightmost juncture is a cut or join. Changing that final decision switches the parity of the number of cars, from even to odd or vice versa.

**Theorem 7, Alternating Row Sums.** \(\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots = 0\), for \(n \geq 1\). In terms of trains, \(T(n + 1, 1) + T(n + 1, 3) + \cdots = T(n + 1, 2) + T(n + 1, 4) + \cdots\).

**Proof.** Given a length \(n + 1\) train \(U\), consider its sequence \(\{u_1, \ldots, u_n\}\) cut / join decisions. Define \(\varphi(U) = T\), another length \(n + 1\) train, by the sequence \(t_i = u_i\) for \(1 \leq i \leq n - 1\) and \(t_n = \overline{u_n}\). (Recall that \(\overline{\text{cut}} = \text{join}\) and \(\overline{\text{join}} = \text{cut}\).) Notice that the number of cars comprising \(U\) and \(T\) differ by one, so that if \(U\) has an odd number of cars then \(T\) has an even number, and if \(U\) has an even number of cars then \(T\) has an odd number. Since \(U \neq T\) and performing \(\varphi\) twice is equivalent to the identity map, \(\varphi\) establishes a bijection between the length \(n + 1\) trains with an odd number of cars and the length \(n + 1\) trains having an even number of cars. \(\square\)
2.3.2 Square Row Sums

Now consider the sum of squares of entries in each row.

\[
\begin{align*}
1^2 + 1^2 &= 1 + 1 = 2 \\
1^2 + 2^2 + 1^2 &= 1 + 4 + 1 = 6 \\
1^2 + 3^2 + 3^2 + 1^2 &= 1 + 9 + 9 + 1 = 20 \\
1^2 + 4^2 + 6^2 + 4^2 + 1^2 &= 1 + 16 + 36 + 16 + 1 = 70
\end{align*}
\]

The sums are the “central binomial coefficients” \( \binom{2n}{n} \) that we saw in the previous section!

**Exploration.** Consider \(1 + 9 + 9 + 1 = 20\). What does this mean in terms of trains? \((T(4, 1))^2 + (T(4, 2))^2 + (T(4, 3))^2 + (T(4, 4))^2 = T(7, 4)\). How do we deal with squares, or even products, of train counts? Concatenation: we’ll basically glue the trains together. The other helpful idea will be to use the Row Symmetry Theorem to make the result more symmetric. Instead of squares, consider

\[
T(4, 1) \cdot T(4, 4) + T(4, 2) \cdot T(4, 3) + T(4, 3) \cdot T(4, 2) + T(4, 4) \cdot T(4, 1) = T(7, 4)
\]

Now the left hand side counts two concatenated trains of length 4 each that have a total of 5 cars and the right hand side counts length 7 trains with 4 cars. Here are the two sets of trains we need to connect.
These sets are very much alike: they each have the same number of trains with the same type of leftmost car and with the same type of rightmost car. The primary difference is that the concatenated trains on the left are 1 too long and all have a cut in the middle juncture. So cut out one half of the car on either side of that middle cut and glue the remainder together, recoloring the fused car as necessary:

![Diagram showing the process of cutting and gluing trains](image)

This gives length 7 trains with 4 cars such that the cars away from the “middle” are unchanged.

For the reverse map, given a length 7 train with 4 cars, break the train between the third and fourth juncture. And one length of train with a cut in the middle and recolor the resulting cars to the left and right as necessary.

**Theorem 8, Square Row Sums.** \( \binom{n}{0}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} \).

In terms of trains, \( T(n+1,1)^2 + \cdots + (T(n+1,n+1))^2 = T(2n+1,n+1) \).

**Proof:** We prove instead

\[
T(n+1,1) \cdot T(n+1,n+1) + \cdots + T(n+1,n+1) \cdot T(n+1,1) = T(2n+1,n+1)
\]

which is equivalent by the Row Symmetry Theorem. Notice that in each product, the sum of the number of cars is \( n + 2 \).

Given \( U \in T(n+1,k+1) \) and \( V \in T(n+1,n+1-k) \), we produce \( W \in T(2n+1,n+1) \) as follows. Let \( U \) be given by the sequence of cut/join decisions \( \{u_1, \ldots, u_n\} \) and \( T \) by \( \{t_1, \ldots, t_n\} \). Define \( W \) by the sequence \( \{u_1, \ldots, u_n, t_1, \ldots, t_n\} \). (Think through how this is equivalent to the cutting and gluing described above.) Since this is a sequence of \( 2n \) cut/join decisions, \( W \) is a length 2\( n + 1 \) train. Since \( U \) has \( k + 1 \) cars, \( k \) of the \( u_i \) are cuts, and since \( T \) has \( n + 1 - k \) cars, \( n - k \) of the \( t_i \) are cuts. Therefore the sequence for \( W \) has \( (n-k) + (k-1) = n \) cuts, so that \( W \) has \( n + 1 \) cars.

For the inverse map, given \( W \in T(2n+1,n+1) \), let \( \{w_1, \ldots, w_{2n}\} \) be its sequence of cut/join decisions. Notice that \( n \) of the \( w_i \) are cuts. Let the length \( n + 1 \) train \( U \) be determined by \( \{w_1, \ldots, w_n\} \), and the length \( n + 1 \) train \( T \) be determined by \( \{w_{n+1}, \ldots, w_{2n}\} \). Since there are \( n \) cuts among the \( w_i \), if there are \( k \) cuts in the \( u_i \), then there are \( n - k \) cuts in the \( t_i \). That is, \( U \in T(n+1,k+1) \) and \( V \in T(n+1,n+1-k) \).

\[
2.3.3 \quad \text{Weighted Row Sums}
\]

Finally, we look at the sum when each \( \binom{n}{k} \) is weighted by the factor \( k \). Since the \( k = 0 \) term contributes 0 to the sum, we start with the \( n = 1 \) row with \( k \geq 1 \).
\[ 1 \cdot \binom{1}{1} = 1 \cdot 1 = 1 \]
\[ 1 \cdot \binom{1}{1} + 2 \cdot \binom{2}{1} = 1 \cdot 2 + 2 \cdot 1 = 4 \]
\[ 1 \cdot \binom{1}{1} + 2 \cdot \binom{3}{2} + 3 \cdot \binom{3}{3} = 1 \cdot 3 + 2 \cdot 3 + 3 \cdot 1 = 12 \]
\[ 1 \cdot \binom{1}{1} + 2 \cdot \binom{4}{2} + 3 \cdot \binom{4}{3} + 4 \cdot \binom{4}{4} = 1 \cdot 4 + 2 \cdot 6 + 3 \cdot 4 + 4 \cdot 1 = 32 \]
\[ 1 \cdot \binom{1}{1} + 2 \cdot \binom{5}{2} + 3 \cdot \binom{5}{3} + 4 \cdot \binom{5}{4} + 5 \cdot \binom{5}{5} \\
= 1 \cdot 5 + 2 \cdot 10 + 3 \cdot 10 + 4 \cdot 5 + 5 \cdot 1 = 80 \]
\[ 1 \cdot \binom{1}{1} + 2 \cdot \binom{6}{2} + 3 \cdot \binom{6}{3} + 4 \cdot \binom{6}{4} + 5 \cdot \binom{6}{5} + 6 \cdot \binom{6}{6} \\
= 1 \cdot 6 + 2 \cdot 15 + 3 \cdot 20 + 4 \cdot 15 + 5 \cdot 6 + 6 \cdot 1 = 192 \]
\[ 1 \cdot \binom{1}{1} + 2 \cdot \binom{7}{2} + 3 \cdot \binom{7}{3} + 4 \cdot \binom{7}{4} + 5 \cdot \binom{7}{5} + 6 \cdot \binom{7}{6} + 7 \cdot \binom{7}{7} \\
= 1 \cdot 7 + 2 \cdot 21 + 3 \cdot 35 + 4 \cdot 35 + 5 \cdot 21 + 6 \cdot 7 + 7 \cdot 1 = 448 \]

\textit{Exploration of Formula.} The sequence 1, 4, 12, 32, 80, 192, 448, \ldots is not as easy to identify as others we have seen. Write this sequence as \( s(1), s(2), \ldots \) Often is helpful to factor the numbers in a sequence, giving special attention to the prime values of \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s(n) )</td>
<td>1</td>
<td>4</td>
<td>12</td>
<td>32</td>
<td>80</td>
<td>192</td>
<td>448</td>
</tr>
<tr>
<td>factors</td>
<td>1</td>
<td>( 2^2 )</td>
<td>( 2^2 \cdot 3 )</td>
<td>( 2^4 \cdot 5 )</td>
<td>( 2^6 \cdot 3 )</td>
<td>( 2^6 \cdot 7 )</td>
<td></td>
</tr>
<tr>
<td>pattern</td>
<td>1 ( \cdot 2^0 )</td>
<td>2 ( \cdot 2^1 )</td>
<td>3 ( \cdot 2^2 )</td>
<td>4 ( \cdot 2^3 )</td>
<td>5 ( \cdot 2^4 )</td>
<td>6 ( \cdot 2^5 )</td>
<td>7 ( \cdot 2^6 )</td>
</tr>
</tbody>
</table>

This leads us to the conjecture that this weighted sum of row \( n \) is \( n \cdot 2^{n-1} \).

\textit{Exploration of Bijection.} What does the \( n = 3 \) statement say in terms of trains? The left-hand side \( 1 \cdot \binom{3}{1} + 2 \cdot \binom{3}{2} + 3 \cdot \binom{3}{3} \) corresponds to \( 1 \cdot T(4, 2) + 2 \cdot T(4, 3) + 3 \cdot T(4, 4) \) where each length 4 train is repeated by a factor related to the number of cars. The right hand side \( 3 \cdot 2^2 \) gives \( 3 \cdot T(3) \), each length 3 train repeated 3 times. We need to find a correspondence between the following two columns of trains.
This is a different situation than we have seen before. How can we distinguish, for instance, between three different light gray cars? For each copy of a length 3 train, add a mark in one of the three positions.

Now the task is to find a way to add marks to the trains corresponding to the left hand side of the equation, and then to find a correspondence between the two columns of marked cars. There are \( k - 1 \) copies of each length 4 train with \( k \) cars. This suggests putting a mark on some \( k - 1 \) of the cars. Since we are looking for a correspondence to marked length 3 trains, we will likely need to remove the last part of the marked length 4 trains, so do not put a mark on the last car. Here is what we have putting a mark at the end of each of the first \( k - 1 \) cars.

For the length 4 trains that end with a white car, the correspondence to length 3 trains is clear enough: remove the last car. This leaves the following trains to be matched up by some rule.
Looking at the marked trains on the left-hand side, we need a way of producing marked cars where the mark is not in the rightmost position. Since we placed a mark at in the last part of a car, we know it is followed immediately by a cut decision, and since we know these remaining trains do not end in a white car, we know the last decision determining the train is join. What we will do is cycle through the sequence of cuts and joins after the mark until the train does end in a white car and then remove it. Cycle by moving the last item of the list to become first. Here is an example of the operation.

\[
\begin{array}{c}
\text{cut, join, cut, join, join} \\
\downarrow \\
\text{join, cut, join, cut, join} \\
\downarrow \\
\text{join, join, cut, join, cut} \\
\end{array}
\rightarrow
\begin{array}{c}
\text{cut, join, cut, join, join} \\
\end{array}
\]

The process is reversible: add a white car and cycle the opposite direction through the cut/join sequence until the mark is followed by a cut decision. In the reverse mapping, cycle by moving the first item of the list to become last. In the previous example, then, simply reverse the arrows.

Here is the correspondence for \(1 \cdot T(4, 2) + 2 \cdot T(4, 3) + 3 \cdot T(4, 4) = 3 \cdot T(3)\) presented twice, once with each side in a “natural” order.

The simple case of trains that end in white cars is actually covered by the same operation: those are the cases where no cycling is required. It’s often a good sign when one operation suffices.

**Theorem 9, Weighted Row Sums.** \(1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + n \cdot \binom{n}{n} = n \cdot 2^{n-1}\).

In terms of trains, \(1 \cdot T(n+1, 2) + 2 \cdot T(n+1, 3) + \cdots + n \cdot T(n+1, n+1) = n \cdot T(n)\).

**Proof:** We produce two sets of marked trains corresponding to each side of the equation and demonstrate a bijection between them.

For the \(k\) copies of each train in \(T(n+1, k+1)\), place a mark in the last position of each of the leftmost \(k\) cars. For the \(n\) copies of each train in \(T(n)\), place a mark in each of the \(n\) possible positions.
Given one of the marked length \( n+1 \) trains, consider the cut / join sequence of decisions at the junctures to the right of the mark. Since the mark is not in the last position, this sequence has length at least 1, and since the mark is not in the rightmost car, this sequence contains at least one cut. Cycle through this sequence, moving the last decision to become the first, until the last decision is cut (this may require no cycling at all). The cycling may change the length of the marked car and leave the mark no longer in the last position of its car. The train now ends in a white car (which does not contain the mark). Removing that rightmost car leaves a length \( n \) containing a mark.

For the reverse map, given a marked length \( n \) train, add a white car at the right end. Now consider the cut / join sequence of decisions at the junctures to the right of the mark. Because the addition white car is to the right of the mark, this sequence has length at least 1, and since the mark is not in the added car, this sequence contains at least one cut. Cycle through this sequence, moving the first decision to become the last (the opposite direction from before), until the first decision is cut (this may require no cycling at all). This produces a length \( n + 1 \) car with a mark in the last position of some car other than the last car.

Notes

[This section includes historical notes, references to articles and books in the bibliography, etc.]

The conjugation defined by reversing cut / join decisions in chapter 2 follows MacMahon [15], whose graphical representation was a sequence of black or white nodes on a line segment.

A second representation he gave uses a “zig-zag” graph related to what is known as the Ferrers diagram of a partition...

The Square Row Sums Theorem, Theorem 8 in chapter 2, is a special case of the following result.

**Vandermonde’s Identity.** \( \sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n} \).

In terms of trains, \( \sum_{k=1}^{n+1} T(r+1,k+1) \cdot T(s+1,n-k+1) = T(r+s+1,n+1) \).

With minor modification, the proof of Theorem 8 in chapter 2 establishes this more general result.
References


Session 3: Figurate Numbers

The sequences:

- Triangle numbers: 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...
- Square numbers: 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
- Pentagonal numbers: 0, 1, 5, 12, 22, 35, 51, 70, 92, 117, ...
- Hexagonal numbers: 0, 1, 6, 15, 28, 45, 66, 91, 120, 153, ...

Some relations, which can be shown algebraically or, more convincingly, visually:

- $1 + 2 + \cdots + n = \frac{n(n+1)}{2} = T(n)$
- $1 + 3 + \cdots + (2n-1) = n^2 = S(n)$
- $1 + 4 + \cdots + (3n-2) = \frac{n(3n-1)}{2} = P(n)$
- $1 + 5 + \cdots + (4n-3) = n(2n-1) = H(n)$
- $T(2n) = 3T(n) + T(n-1)$
- $T(2n+1) = 3T(n) + T(n+1)$
- $S(n) = T(n) + T(n-1)$
- $P(n) = S(n) + T(n-1)$
- $H(n) = P(n) + T(n-1)$
- $3P(n) = T(3n-1)$
- $H(n) = 3T(n-1) + T(n) = T(2n-1)$

Figure 1: Three times a pentagonal number is a triangular number.
Session 4: Restricted Trains

Notes for chapter three of *Hands-On Combinatorics*.

Fibonacci numbers: recurrence relation $F_n = F_{n-1} + F_{n-2}$ with initial values $F_0 = 0$, $F_1 = 1$. Next few terms 1, 2, 3, 5, 8, 13, 21, 34, 55, 89.

We consider several types of restricted trains, where only certain kinds of cars are allowed as components. The notation $T_R(n)$ indicates the number of length $n$ trains with restriction $r$. Here we will consider three restrictions: only length 1 and 2 cars are allowed, only odd length cars are allowed, and no length 1 cars are allowed. Notation $T_{1,2}(n)$, $T_{2k+1}(n)$, and $\hat{T}_1(n)$, respectively. (The “hat” notation is common for something not in a list.)

**Just 1s & 2s**

\[ T_{1,2}(n) = F_{n+1} \]

Sketch: Build length $n$ trains by adding a white 1 to the end of all length $n-1$ trains and a red to the end of all length $n-2$ trains; $T_{1,2}(n) = T_{1,2}(n-1) + T_{1,2}(n-2)$. For initial values, $T_{1,2}(2) = 2 = F_3$ so $T_{1,2}(1) = 1 = F_2$, thus the offset in the formula from $n$ to $n+1$.

**Odd length cars**

\[ T_{2k+1}(n) = F_n \]

Sketch: Build length $n$ trains by adding a white 1 to the end of all length $n-1$ trains and extending the last car of all length $n-2$ trains by two (which
keeps the car an odd length); \(T_{2k+1}(n) = T_{2k+1}(n - 1) + T_{2k+1}(n - 2)\). For initial values, \(T_{2k+1}(1) = 1 = F_1\) and \(T_{2k+1}(2) = 1 = F_2\).

\[T_{\hat{1}}(n) = F_{n-1}\]

Sketch: Build length \(n\) trains by extending the last car of all length \(n - 1\) trains and adding a red 2 to the end of all length \(n - 2\) trains by two; \(T_{\hat{1}}(n) = T_{\hat{1}}(n - 1) + T_{\hat{1}}(n - 2)\). For initial values, \(T_{\hat{1}}(2) = 1 = F_1\) and \(T_{\hat{1}}(3) = 1 = F_2\), thus the offset from \(n\) to \(n - 1\).
Connections

We have three different train manifestations of the Fibonacci numbers. How do they connect to each other?

1s & 2s and Odds

\[ T_{1,2}(n) = T_{2k+1}(n + 1) \]

By definition, an odd number can be written as \( j = 2k+1 \). We will exchange the car of length \( j \) for \( k \) reds followed by a white. But what about trains of 1s&2s that end in a red? We add a white at the end. That takes care of the offset.

1s & 2s and No 1s

\[ T_{1,2}(n) = T_{\hat{1}}(n + 2) \]

In general, a train consisting of 1s and 2s has many cuts, while a train with no 1s has fewer. The connection here is to toggle cuts and joins. To make sure this does not a 1 on the either end of the new train, put a join at the beginning and end.
This could be done directly, but it is enough to “go through” the 1s & 2s manifestation of length \( n - 1 \) trains.

**Fibonacci Identities**

There are many identities about Fibonacci numbers. In looking for a combinatorial proof, different manifestations of Fibonacci numbers are more conducive.

**Sums**

\[ F_1 + \cdots + F_n = F_{n+2} - 1 \]

Consider the no 1s manifestation of the Fibonacci numbers. We show \( T_2(2) + \cdots + T_2(n+1) = T_1(n+3) - 1 \). Given a train on the left hand side, add a single car of the appropriate length to make a length \( n + 3 \) train. On the right hand side, the train consisting of a single length \( n + 3 \) car is missing, giving the minus 1 term in the identity.

\[ F_1 + F_3 + \cdots + F_{2j+1} = F_{2j+2} \]

Since the summand indices differ by two, the odd length car manifestation suggests itself. We show \( T_{2k+1}(1) + T_{2k+1}(3) + \cdots + T_{2k+1}(2j+1) = T_{2k+1}(2j+2) \). The difference in length for each train on the left hand side and the desired length \( 2j + 2 \) is an odd number; simply add a car of that length to each train.

[[Figure missing.]]
Products

Looking at squares of Fibonacci numbers and adding consecutive pairs leads to the identity

\[ F_n^2 + F_{n+1}^2 = F_{2n+1} \]

Here we use the just 1s & 2s manifestation, so we prove \( T_{1,2}^2(n-1) + T_{1,2}^2(n) = T_{1,2}(2n) \). For terms like \( T_{1,2}^2(n) \), we consider all possible concatenations of two length \( n \) trains. The concatenation juncture is marked by a heavy line.

The trains corresponding to \( T_{1,2}^2(n) \) are already the correct length. Notice that these all have a cut in the middle position. The \( T_{1,2}^2(n-1) \) are two too short, and certainly there are length \( 2n \) trains that have a join in the middle position. For these shorter trains, insert a red 2 between the two halves.

Careful consideration of the proof shows that this reasoning applies to any juncture, not just the middle. The same proof gives the following result, which would be harder to guess.

\[ F_mF_n + F_{m-1}F_{n-1} = F_{m+n-1} \]

Bulgarian Solitaire

One-per-car operation from stacked representation of trains (similar to Ferrers’ diagrams of partitions), number of Garden of Eden states is \( 2^{n-1} - F_{n+1} \).
Acme Integral Weights Exchange International

You got a job at the prestigious AIWEI, which uses balances to identify rocks that weigh integer numbers of ounces.

1A You start with a beginner balance that only allows weights in one pan, opposite the rock you want to weigh. You will be given rocks of arbitrary weight—your job is to identify which ones weigh an integer number of ounces: 1 oz., 2 oz., etc. The office will give you any four weights you want: Which ones do you choose? What weights can you determine?

1B You move up to quality control, using the same balance. Now you are given rocks known to have integer weights—your job is to determine what integer. Can you do better than before? Which four weights do you want now? What weights can you determine?

2A As a reward for your great work, you are given a balance that allows weights in both pans. Again, you will be given rocks of arbitrary weight, and your job is to pick out the ones which weigh an integer number of ounces. Which four weights do you want now? What weights can you determine?

2B Finally, you are in quality control with the state-of-the-art balance that allows weights in both pans. As before, you are given rocks known to have integer weights, and your job is to determine what integer. Now which four weights do you want now? What weights can you determine?
Session 5: Balancing, AIWEI solutions

The requirement of “no gaps” is purposely left off of the worksheet instructions to allow for refining the problem.

Weights on one side

Four weights allows for, at most, 15 distinct total weights: each of the rocks has two potential states, on or off the balance, and \(2^4 = 16\)—then we exclude the “all off” option since there are no weight 0 rocks to worry about. (One can also arrive at 15 by adding up binomial coefficients: 4 ways to use a single weight, 6 ways to use two of the four weights, 4 ways to use three weights, and 1 way to use all four weights.)

Weights have to be carefully chosen to achieve 15 distinct values. For instance, asking for 2, 3, and 5 ounce weights is redundant since that makes two different ways to counterbalance a five ounce rock. Choosing very different weights, such as 1, 10, 100, 1000, guarantees no redundancy, but leaves many gaps.

For situation 1A, you need to identify which rocks have integer weights. To achieve no gaps in rock weights you can determine, work from the bottom up. To weight a 1 ounce object, a 1 ounce weight is necessary. To weigh a 2 ounce object, a 2 ounce weight is needed, and now you can also weigh a 1+2 = 3 ounce object. The next necessary weight is 4 ounces, which also allows determination of 4 + 1 = 5, 4 + 2 = 6, and 4 + 2 + 1 = 7 ounce objects. The last weight then is 8 ounces, which also allows for objects weighing 9, . . . , 15 ounces—achieving the maximal number of weights with no gaps.

For situation 1B, you know that the rocks you weigh have integer weights, just not which integer. This allows you to do more: you do not need a 1 ounce weight, for instance, since a rock that weighs less than 2 ounces must in fact weigh 1 ounce. In this situation, order 2, 4, 8, and 16 ounce weights, which allow you to determine weight 1 to 30 ounce rocks. (You cannot be sure of a 31 ounce rock, you can only say it weighs more than 2 + 4 + 8 + 16 = 30 ounces.)

Weights on both sides

The essential difference in allowing weights on both pans is subtraction. For instance, with 1 and 3 ounce weights, a 2 ounce rock can be determined by putting it and the 1 ounce weight opposite the 3 ounce weight.

For the total number of possible weights that can be measured, each of the four weights you order now has three possible states: in the pan with the rock to be weighed, in the other pan, or off. As before, that suggests \(3^4 - 1 = 80\) possibilities. But some of these do not make sense in the context of the situation, as you would never put a larger total of weights in the pan with the rock (rocks have positive weights). For every possible pair of weights in the two pans, one has a smaller total with the rock. That means there are \(80/2 = 40\) possibilities.
(Another way of getting 40, from binomial coefficients. The largest weight used must be in the pan opposite the object, but any smaller weights can go in either pan. That gives

\[
\binom{4}{1} + \binom{4}{2} \cdot 2 + \binom{4}{3} \cdot 2^2 + \binom{4}{4} \cdot 2^3 = 4 + 12 + 12 + 8 = 40
\]

possible weighings.)

Now proceed as before. In situation 2A, to weight a 1 ounce object, a 1 ounce weight is necessary. Now the 1 can be added to or subtracted from a heavier weight (by going in the same or opposite pan), so we do not need a 2 ounce weight to measure a 2 ounce object. Rather, we can move up to a 3 ounce weight, since “1 ounce weight + 2 ounce object = 3 ounce weight.” The 1 and 3 ounce weights also determine weights of 3 and 4 ounce objects. To weigh a 5 ounce object, a 9 ounce weight is enough, since 9 - 4 = 5. Now you can also measure weights 6, \ldots, 13. The fourth weight should be 27 ounces, allowing for 27 - 13 = 14, 15, \ldots, 27 + 13 = 40—again, achieving the maximal number of weights with no gaps.

Situation 2B is again a doubling argument: weights 2, 6, 18, 54 allow determining known integer weight rocks of 1, \ldots, 80 ounces. Groups have tried further generalizations where one could determine, for instance, whether a rock was closer to 1 or 4 ounces.
Session 7: Daisy

- Setup: 13 stones in a tight circle (so that adjacent stones touch)
- Legal moves: in each turn, take any one stone or any two touching stones
- Winner: takes the last stone

Although this seems very much like “take 1 or 2” with a pile of 13 stones, the adjacency requirement for taking pairs makes it a very different game. For any number of stones, in fact, the second player has a simple winning strategy.

The first player takes one or two adjacent stones from the circle. The second player should remove one or two stones at the opposite side of the circle in order to leave two isolated arcs having the same number of stones. (For the 13 game, in response to 1 or 2 being removed, player 2 should take the opposite 2 or 1, respectively, leaving two arcs of 5 stones each.) Then, whatever player 1 does in one arc, player 2 repeats in the other arc, as if there were a mirror on the board. This guarantees that player 2 will make the last legal move. In combinatorial games, this is called “strategy stealing.”

While that is a complete analysis of optimal strategy, in actual play it may be of interest to know how to win should player 2 not use this approach. For notation, we will write 12 for the daisy after one petal is removed leaving 12 adjacent petals. After player 1 moves, the game is 12 or 11. The strategy described for player 2 above is to then move to 5 + 5. There are additional winning moves for player 2 from 11: 1 + 8, 2 + 7, and 3 + 6, although how to proceed in each case is not as easily described.

The following table gives the moves that Player 1 should make if Player 2 does not leave 5 + 5, 3 + 6, 2 + 7, or 1 + 8 (if presented with one of these, player 1 should remove something from the larger pile, probably, and hope that player 2 doesn’t have a strategy). Play is presented down to a “mirror” state, where strategy stealing may be employed, or 1 + 2 + 3, which is another winning state (verify this: in two turns you can leave your opponent with 2 + 2, 1 + 1 + 1 + 1, or 1 + 1).

<table>
<thead>
<tr>
<th>Player 2 from 12 or 11</th>
<th>Player 1</th>
<th>next Player 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>11 or 5, 6</td>
<td>5, 5</td>
<td></td>
</tr>
<tr>
<td>4,7; 3, 7; 2, 9 or 2, 8</td>
<td>2, 7</td>
<td>3, 3; 1, 2, 3, or 1, 1, 2, 2</td>
</tr>
<tr>
<td>4, 6 or 4, 5</td>
<td>4, 4</td>
<td></td>
</tr>
<tr>
<td>3, 8</td>
<td>2, 3, 4</td>
<td>3, 3; 1, 2, 3 or 1, 1, 2, 2</td>
</tr>
<tr>
<td>1, 10 or 1, 9</td>
<td>1, 2, 6</td>
<td>1, 2, 3 or 1, 1, 2, 2</td>
</tr>
</tbody>
</table>

Notice that after the first move, the circular aspect of the game is irrelevant—the remaining stones might as well be in a row rather than around a circle. Starting with stones in a row is the next game.
Kayles

- Setup: 1 stone, a space, and 11 stones in a tight row (so that adjacent stones touch); as a preliminary version, you may begin with 13 stones in a tight row
- Legal moves: in each turn, take any one stone or any two touching stones
- Winner: makes the last move

The name kayles comes from a 14c. English slaughter of the French *quilles*, the term for bowling pins. The bowling idea is that you can knock over one pin or two pins that are next to each other. In the preliminary version with 13 stones in a row, the first player can win by strategy stealing: remove the middle pin, leaving two sections of 6 adjacent pins, and mirror whatever your opponent does.

To make the game more interesting, start with the second pin missing, so 1 alone and a row of 11. (This is basically the misère version of 11 in a row.) Now the strategy is surprisingly complicated. Below is a “brute force” solution of this case, and computer work has been done up to very large numbers of pins, but there is no nice general analysis.

The first player can win, although it requires a very specific move: take either the 4th or 8th pin of the 11, leaving 1 + 3 + 7 (see the Daisy page for notation). The second player has 11 possible choices, which will require 4 possible responses from the first player. The following table outlines all possibilities down to “mirror” states where strategy stealing may be employed, or 1 + 2 + 3, another winning state.

<table>
<thead>
<tr>
<th>Player 2 from 1, 3, 7</th>
<th>Player 1</th>
<th>next Player 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3, 7 or 1, 2, 7</td>
<td>2, 7</td>
<td>3, 3; 1, 2, 3, or 1, 1, 2, 2</td>
</tr>
<tr>
<td>1, 2, 3, 4</td>
<td>2, 3, 4</td>
<td>3, 3; 1, 2, 3 or 1, 1, 2, 2</td>
</tr>
<tr>
<td>1, 1, 1, 7</td>
<td>1, 1, 1, 2, 3</td>
<td>four 1s; six 1s; 1, 2, 3 or 1, 1, 2, 2</td>
</tr>
</tbody>
</table>

1, 3, 6; 1, 3, 5; 1, 1, 7;
1, 2, 3, 3; 1, 3, 3, 3;
1, 1, 3, 5 or 1, 1, 3, 4 | 1, 1, 3, 3 |

In actual play, it is also helpful to know how to win as the second player should your opponent not leave you with 1 + 3 + 7.

<table>
<thead>
<tr>
<th>Player 1 from 1, 11</th>
<th>Player 2</th>
<th>next Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>11 or 1, 5, 5</td>
<td>5, 5</td>
<td></td>
</tr>
<tr>
<td>1, 10; 1, 9; 1, 2, 8 or 1, 2, 7</td>
<td>1, 2, 6</td>
<td>1, 2, 3 or 1, 1, 2, 2</td>
</tr>
<tr>
<td>1, 4, 6 or 1, 1, 9</td>
<td>1, 1, 4, 4</td>
<td></td>
</tr>
<tr>
<td>1, 4, 5</td>
<td>1, 2, 2, 4</td>
<td>1, 1, 2, 2</td>
</tr>
<tr>
<td>1, 3, 7</td>
<td>3, 7; 1, 2, 3, 4</td>
<td>hope for opponent error</td>
</tr>
<tr>
<td>1, 3, 6 or 1, 1, 8</td>
<td>1, 1, 3, 3</td>
<td></td>
</tr>
</tbody>
</table>
Session 8: Cuisenarea

Rather than using Cuisenaire rods to make “trains” which are basically one-dimensional, one can ask how many ways Cuisenaire rods can fill out two-dimensional, three-dimensional, etc. regions.

The numbers get large very quickly, so it helps to use narrow regions, e.g., $2 \times n$ strips.

• How many ways can you tile a $2 \times n$ strip using just reds? (Here, the dominoes can be horizontal or vertical.)

The answer is the Fibonacci sequence. To see the connection, cover the top half on the region to create a train tiled by reds and whites (bottom halves of vertical reds).

• How many ways can you tile a $2 \times n$ strip using white squares and L’s? (L’s are three-square pieces that have four possible orientations.)

Given a tiled $2 \times n$ strip, where is the rightmost full vertical break through both rows?

– If the vertical break is one position in from the righthand edge, the last column must be two white squares.
– If the last vertical break is two positions in from the righthand edge, the last two columns must consist of an L and one square, in four possible orientations.
– If the last vertical break is three positions in from the righthand edge, the last three columns must consist interlocking L’s, which can occur in two ways.

Convince yourself that there are no other possibilities for the rightmost vertical edge, that is, it must at most three positions in from the righthand edge. So the number of tilings $a_n$ is given by a three term recurrence (compare to the two term Fibonacci recurrence), specifically

$$a_n = a_{n-1} + 4a_{n-2} + 2a_{n-3} \quad a_1 = 1, a_2 = 5, a_3 = 11, a_4 = 33$$

In the wonderful resource Online Encyclopedia of Integer Sequences, this is A127864.

• How many ways can you tile a $2 \times n$ strip using just whites and reds?

This one is trickier. The rightmost full vertical break can be arbitrarily far back. Instead, look at what happens one position from the righthand edge: there can be a full vertical break, a break through one row but not the other, or a break in neither row.

Let $b_n$ be the number of tilings for a $2 \times n$ strip.
If there is a full vertical break one position from the righthand edge, there are two possibilities for the last column—two whites or a vertical red. So there are \(2b_{n-1}\) such tilings of the \(2 \times n\) strip.

If there are breaks in neither row one position from the righthand edge, then the last two columns must consist of two horizontal reds. There are \(b_{n-2}\) such tilings.

That leaves \(b_n - 2b_{n-1} - b_{n-2}\) tilings with a break through exactly one row one position from the righthand edge. Of these, \(2b_{n-2}\) have a horizontal red and two whites in the last two columns (the coefficient 2 since there are two choices for the row of the red).

So there are \(b_n - 2b_{n-2} - b_{n-3}\) tilings for a \(2 \times (n-1)\) strip with a break through exactly one row one position from the righthand edge. That is, the last column is a white square and half of a horizontal red. This can be extended in one way to a tiling of a \(2 \times n\) strip: in one row, extend the white into a red; in the other row, add a white at the end of the red.

This gives the three term recurrence

\[
b_n = \begin{cases} 
2b_{n-1} & \text{full break} \\ \ no \ break \ & \ + b_{n-2} \\ \ \text{half break} \ & \ + 2b_{n-2} + (b_{n-1} - 2b_{n-2} - b_{n-3}) \\
\end{cases} 
= 3b_{n-1} + b_{n-2} - b_{n-3}
\]

with \(b_1 = 2, b_2 = 7, b_3 = 22, b_4 = 71, \ldots\) In the OEIS, this is A030186.

- How many ways can you tile a \(2 \times n\) strip using any rods?

This is more complicated yet; the recurrence goes all the way back for each new term. If there are no vertical reds, then the tiling is two rows of trains, so \(2^{2(n-1)}\). The presence of vertical reds leaves smaller areas that are two rows of trains. The first few values, initially grouped by number of vertical reds, are

\[
c_1 = 2^0 + 1 = 2 \\
c_2 = 2^2 + 2 \cdot 2^0 + 1 = 7 \\
c_3 = 2^4 + (2 \cdot 2^2 + 1 \cdot 2^0 \cdot 2^0) + 3 \cdot 2^0 + 1 = 29 \\
c_4 = 2^6 + (2 \cdot 2^4 + 2 \cdot 2^2 \cdot 2^0) + (3 \cdot 2^2 + 3 \cdot 2^0 \cdot 2^0) + 4 \cdot 2^0 + 1 = 124
\]

and I worked out \(c_5 = 533\). This matches OEIS A052961, which has recurrence relation

\[
c_n = 5c_{n-1} - 3c_{n-2}
\]

but the citation does not mention tiling. Exercise: Explain why this recurrence matches the tiling enumeration.
Also, what about filling a $2 \times 2 \times 2$ cube with reds? (Here, think of reds as having dimension $1 \times 1 \times 2$ in three possible orientations.)

Consider the cube positions as 000, 001, 010, 011, 100, 101, 110, and 111, shorthand for the eight $(x, y, z)$ coordinates of the corners.

The origin must be connected to an adjacent positions, 001, 010, or 100. We list all tilings that include a 000-001 red.

000-001, 010-011, 100-101, 110-111
000-001, 010-011, 100-110, 101-111
000-001, 010-110, 011-111, 100-101

By symmetry, there are three tilings starting from each of the 000-010 and 000-100 dominoes, giving a total of 9.

Such tilings have application in theoretical physics. For example, statistical mechanics is concerned with ways that “dimers” can cover rectangular grids of atomic positions.
Session 9: Frogs & Toads

Set-up: On a strip with $2k + 1$ positions, there is a empty space in the middle, $k$ frogs on one side, $k$ toads on the other. The goal is for the animals to swap positions. There are two legal moves: an animal can slide forward one position into an empty space, and an animal can jump forward over one of the other sort of animals into an empty space. What is the minimal number of moves for the total swap?

- T.F slide
- TF jump
- F.T slide

For $k = 1$, three moves suffice, which we can denote as SJS – what is ambiguous about that notation, and why is it OK? For $k = 2$, we worked out SJSJJSJS.

**Theorem 2.1.** Swapping the animals requires exactly $k^2 + 2k$ moves.

**Proof.** Each animal moves $k + 1$ positions, for a total of $2k(k + 1) = 2k^2 + 2k$. Each slide moves one position, a jump two positions. We need to show that moving a total of $2k^2 + 2k$ positions requires exactly $k^2 + 2k$ moves.

In order to switch sides, each of $k$ toads must jump over or be jumped over by each of $k$ frogs. Because of the direction of jumps, there cannot be more than these $k^2$ jumps, so we conclude that there are exactly $k^2$ jumps.

Those $k^2$ jumps account for $2k^2$ of the total chance of position. That leaves $2k$ is the position total, which must be accomplished by $2k$ slides. Therefore the total number of moves is $k^2 + 2k$, jumps plus slides. \( \square \)

One can prove a more specific sequence of jumps and slides in a recursive way that uses previous sequences. For example, the $k = 2$ sequence includes two $k = 1$ sequences: (SJS)JJ(SJS).
Session 10: Exploring Recursion with the Josephus Problem or how to play “one potato, two potato” for keeps

Summary

The Josephus problem is addressed in many discrete mathematics textbooks as an exercise in recursive modeling, with some books (e.g., Ensley & Crawley, Discrete Mathematics: Mathematical Reasoning and Proof with Puzzles, Patterns and Games, John Wiley 2006, and Graham, Knuth, Patashnik Concrete Mathematics, Addison-Wesley 1994.) even using it within the first few pages as an introductory problem to intrigue students. Since most students are familiar with the use of simple rhymes (like Eeny-meeny-miney-moe) for decision-making on the playground, they are comfortable with the physical process involved in this problem. For students who may wish to pursue this topic independently, Herstein & Kaplansky Matters Mathematical, Chelsea 1974 and Schumer “The Josephus problem: Once more around,” Mathematics Magazine 75 (2002) 12-17 provide nice surveys and bibliographies, and the website http://webspace.ship.edu/~deensley/DiscreteMath/ provides web-based tools for exploring the problem directly. The activities presented here are intended to be completed by students in a single class period early in the semester. We find that an opening student-centered problem can get the class involved and set a good tone for the semester. Moreover, we find that many issues arising from this particular problem can be built upon throughout the course. The next section provides some suggestions for connections to other parts of the course.

Notes for the instructor

The Josephus problem can be explored through role playing or through carefully constructed pencil and paper activities, depending on the amount of time one wishes to devote to it. We list below some of the things we discuss just before the activity as well as some of the contexts in which we have students revisit the problem later on.

• A good preliminary discussion on recursion can be initiated with the following problems.

1. Pose the question, “What is \(1 + 2 + 3 + \cdots + 19 + 20?\)” This provides a good opportunity to share the creative idea of regrouping in order to sum 10 copies of 21 for a total of 210.

2. Followup with the question, “What is \(1 + 2 + 3 + \cdots + 20 + 21?\)” Some students will try the regrouping trick, but at least one should point out that you can simply add 21 to the previous answer.

3. This idea of using a “similar but simpler” problem that has been solved previously is the very essence of recursive thinking.
The activities presented here have been written to be completed with paper and pencil, but with the investment of more time one can have students act out the roles. This is a good ice-breaking activity early in the semester, but it does take more time. Through role playing, students will discover for themselves issues like “We need to remember who was first,” and “We need a system for describing who is the last one left.”

There will be several opportunities later in a discrete mathematics course when one can reprise the Josephus game as a source for exercises and motivational examples. Computer science courses often use this problem as an exercise in recursive programming or in maintaining circular linked lists. Hence, with some cooperation from a friendly computer science instructor, this problem can prove useful in more than one context.

1. (Mathematical induction) In the Josephus problem with skip number 2, prove that for all integers $n \geq 0$, if the game starts with $2^n$ players, then the person in position 1 will be the last person left. (This uses induction with the induction step involving the one pass all the way around the circle for the first time in which the even numbered people are eliminated.)

2. (Follow up) In the Josephus problem with skip number 2, if $0 \leq k < 2^n$ and the game starts with $2^n + k$ players, then the person in position $2k + 1$ will be the last person left. This is a non-inductive argument consisting of removing the first $k$ (even numbered) people and then applying #1 to the remaining circle of size $2^n$.

3. (Binary representation of numbers) Define the cyclic left shift of a binary numeral $b$ as the number obtained from shifting the leading (i.e., leftmost) 1 bit to the rightmost end of the numeral. For example, the cyclic left shift of the binary numeral 1001101 is the numeral 0011011, which is the same as 11011. Show that if $0 \leq k < 2^n$, then the cyclic left shift of the binary representation of $2^n + k$ is the binary representation of $2k + 1$. Hence, the cyclic left shift of a number $m$ gives the last person left in the $m$ person Josephus game with skip number 2. This gives an “application flavor” to the study of binary numbers that may make them more intriguing.

4. (Modular arithmetic) When introducing modular arithmetic, an analogy can be made to the Josephus problem in which the original circle of people are numbered 0 through $n - 1$. In particular, the patterns within the tables of “last person left” all have the relationship “add k” but with the provision that the addition “wraps around the circle” to refer to the actual people.
Worksheet on Exploring recursion with the Josephus Problem (Or How to play “One Potato, Two Potato” for keeps)

Introduction

Ancient mathematics problems that still hold their own are always fun to play with. A particularly good one, which happens to be named for a first century historian, has its origins in the Jewish - Roman war. The historian Flavius Josephus was apparently trapped by the Romans in a cave with 40 fellow Jewish rebels. As good soldiers they decided on suicide rather than capture, so they formed a circle and agreed that every third person would be killed until no one was left.

Josephus and a friend were more keen on being captured than their colleagues, so they quickly found the spots to stand to ensure they were the two remaining at the end of the grisly proceedings. Hence, the mathematically inept suffered an untimely demise while Josephus and his friend lived to tell the tale.

This morbid story doesn’t seem like much of a game or puzzle, but it has the same basic structure (with terminal consequences) as the age old way of choosing someone from a group: the “one potato, two potato” algorithm. We will spend some time in class today playing this type of game and analyzing our results.

Analyzing the Josephus Problem

In general, when we play the “Josephus game,” there will be a certain number of people standing in a circle, and a “skip number” that tells us how many people to count before removing someone from the circle. In the classical example described above, the number of people is 41 and the skip number is 3.

Let’s look at a simpler example. This time, there will be only six people in the circle, but we will keep the skip number at 3. We’ll continue to play until there is only one person remaining. Let’s say the people, named Ann, Beth, Chris, Dave, Emma, and Fred, are arranged as shown in Figure 2.

Figure 2: Six people in a circle

In this case, we decide to start counting with Ann. We count Ann and Beth, and when we get to the third person, Chris, he is removed from the circle. With
Chris gone, we continue counting with Dave. We count Dave and Emma, and when we get to Fred, he is eliminated from the circle. Now there are four people left in the circle: Ann, Beth, Dave, and Emma, and the counting continues with Ann. We count Ann and Beth, and then Dave is eliminated. The current situation is displayed in Figure 3.

![Figure 3: Three people remain](image)

We next count Emma and Ann, and remove Beth, and the counting once again continues with Emma. We count Emma, Ann, and then Emma is removed, so Ann is the person who is left standing at the end.

An important thing to notice about this process is that we need to know which person to start the counting with at each step, including the first step. If we remove a couple of people and then go on a coffee break, we might come back and forget who to resume the counting with.

For discussion: Can you think of a way that we could remember which person we need to start the counting with at each step?

One solution is for us to put a funny hat on the person we need to start the counting with at each step. In our diagrams, we will put a thick circle around the “starting person.”

Let’s try the game again, this time with seven people (named A, B, C, D, E, F, and G) and removing every fifth person. Recall that we say that the “skip number” is equal to 5. Figure 4 shows diagrams illustrating how such a game progresses. Note that the players are removed in the order E, C, B, D, G, A, and person F is the last one standing.

Exercise 1. On your own, play the Josephus game with $n$ players and a skip number of $k$ for each of the following values. Determine who is the last person standing.

- $n = 6, k = 2$
- $n = 10, k = 3$
- $n = 11, k = 3$

Changing the Starting Player

What happens if we decide to keep the values of $n$ and $k$ the same, but change the person we start the game with? How does this affect the outcome? Let’s go back to the example with seven people and a skip number of 5. Let’s say the people are named Terry, Ursula, Vivian, Walter, Xander, Yolanda, and Zack, and we want to start the game with Walter as Figure 5 shows.
Start with A to be counted first

E was removed and F is to be counted first

C was removed and D is to be counted first

B was removed and D is to be counted first

D was removed and F is to be counted first

G was removed and A is to be counted first

**Figure 4: The game with seven people**

**For discussion:** In the game shown on the left in Figure 5, F is the last person standing. Who will be the last person standing in the game shown on the right? Can you figure it out without playing the entire game again?

*If you said that Ursula would be the last person standing, you are correct! When we have seven people and a skip number of 5, the last person standing is the sixth one around the circle from the starting player. (Here we count the starting person as the first player around the circle.) In mathematical notation, we will write this as $J(7, 5) = 6$. The $J(n,k)$ notation is very handy for describing the last person left in the Josephus game that starts with $n$ people in the circle and eliminates every $k^{th}$ one. For example, the result of our first example can be described by simply writing $J(6, 3) = 1$.**

**Exercise 2.** Go back to the three games you played in Exercise 1. Using the mathematical notation we have defined, find the value of $J(n,k)$ in each of the following cases.

   a. $J(6, 2)$  
   b. $J(10, 3)$  
   c. $J(11, 3)$
Recursion: Using What Came Before

This idea of changing the starting player can be very helpful for finding patterns in the Josephus problem. Consider the game with eight people and a skip number of 5, as shown in Figure 6. After the first step of this game, E is eliminated from the circle, and we have the situation in Figure 7.

Now what? Well, we continue to play the game as before, or we might notice that we have seen this situation before. This is a game with seven players and a skip number of 5. We already determined that the last person standing in this game is the sixth person around the circle from the starting player. In this case, that means that C is the last person standing.
For discussion: Finish playing the game to verify that C is the last one standing.

Here is another example of this idea. In Exercise 2(c), you determined that \( J(11, 3) = 7 \). That is, in a game with eleven people and a skip number of 3, the seventh person around the circle from the starting person will be the last one standing. How does this help us determine the value of \( J(12, 3) \)? Consider the first step of the game with twelve people and a skip number of 3. The first person eliminated is person 3, and person 4 becomes the new starting player. Now there are eleven people remaining, and we know that the last one standing will be the seventh person around the circle starting with person 4. This is person 10. You can verify on your own that \( J(12, 3) = 10 \).

Finding the Pattern

There is a pattern to how the position of the last survivor changes as we change the number of people initially standing in the circle. To see this pattern, we need to experiment and compute the answer for many different examples. In the table in Exercise 3, the top row shows the number of people in the circle, and the bottom row shows the position of the last person standing when the skip number is 3. The values we have already determined are filled in for you.

Exercise 3. Fill in the rest of the table, either by playing each game or by appealing to the “using-what-came-before” strategy.

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J(n, 3) )</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>7</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. What pattern do you notice in the table?

2. Can you explain in terms of the “using-what-came-before” strategy why this pattern holds?

3. On your own, make a similar table but change the skip number to 4. Can you predict what pattern you will see?

An easier variation

A game that’s a little better suited for detailed analysis is the variation where every second person is eliminated — that is, the skip number is 2. The game will officially be played with people named 1, 2, \ldots, n in a circle (with the numbers going clockwise). We go around the circle clockwise getting rid of every second person (Person 2 is the first to go) until no one is left. For example, if we start with four people, then the people are eliminated in the order 2, 4, 3, 1, so person 1 is the last survivor.

We will let \( J(n) \) denote the last survivor in the game which starts with \( n \) people and has a skip number of 2. (That is, we use \( J(n) \) instead of \( J(n, 2) \).)
Exercise 4. Fill in the rest of the table, either by playing each game or by appealing to the “using-what-came-before” strategy.

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( J(n) )</td>
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<td></td>
<td></td>
<td></td>
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<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>n</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
<th>25</th>
<th>26</th>
<th>27</th>
<th>28</th>
<th>29</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J(n) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

1. How is the value of \( J(n) \) related to the value of \( J(n - 1) \)?
2. What will be the next value of \( n \) for which \( J(n) = 1 \)?
3. How would you describe a formula for \( J(n) \) that would allow someone to quickly figure out the last place in line given any \( n \)?

Josephus and his buddy

In the original story, Josephus actually escapes with a friend, so in reality he had to know the positions of the last two survivors of this macabre game. To keep it simple, let’s still use the game with skip number 2, but now we will use \( F(n) \) to denote the required position of the friend in the Josephus game starting with \( n \) people.

Exercise 5. Play the Josephus game (with every second person eliminated, as above) for various \( n \) and record the numbers \( J(n) \) and \( F(n) \) of the last person alive and of the next-to-the-last person alive, respectively. Find more values than in the table below if you think it is helpful to do so. Remember to try to use things you already know as you tackle larger and larger values of \( n \).

<table>
<thead>
<tr>
<th>n</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J(n) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>( F(n) )</td>
<td></td>
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<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. How is the value of \( F(n) \) related to the value of \( F(n - 1) \)?
2. What will be the next value of \( n \) for which \( F(n) = 1 \)?
3. Is there a direct relationship between \( J(n) \) and \( F(n) \)?

Further questions for exploration

The following problems, as well as the ones above, can be explored with the applet found under Section 1.1 on the website

http://webspace.ship.edu/~deensley/DiscreteMath/flash/
Exercise 6. Fill in the following table using the “One potato, two potato” game on \(n\) people, starting the first “one potato” on person 1. For those not familiar with this method of choosing a person on the playground, this is simply the Josephus problem with every eighth person eliminated. That is, in the table below we use \(P(n)\) to mean the same thing as \(J(n, 8)\) from the previous discussion.

<table>
<thead>
<tr>
<th>(n)</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(n))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. If the ______ students in this class stand in a circle in alphabetical order and do “one potato, two potato”, who will be the last person left?

2. Suppose in the game with 6 people, Josephus is person 1 but before the game starts, the Roman leader says, “Hey Joey, you pick the skip number.” What should he say so that he is the last person left?

3. Is it possible for Josephus to always come up with a response to the previous question no matter how many people are originally in the circle?
Solutions

Exercise 1. We will use the conventions of labeling the people A, B, C, etc. clockwise around the circle and starting our count with person A.

1. For \( n = 6 \) and \( k = 2 \), the last person left is E.
2. For \( n = 10 \) and \( k = 3 \), the last person left is D.
3. For \( n = 11 \) and \( k = 3 \), the last person left is G.

Exercise 2.

1. \( J(6, 2) = 5 \)
2. \( J(10, 3) = 4 \)
3. \( J(11, 3) = 7 \)

Exercise 3. Here is the completed table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J(n, 3) )</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>10</td>
<td>13</td>
<td>2</td>
</tr>
</tbody>
</table>

1. For all \( n \geq 2 \), person \( J(n, 3) \) is three more around (clockwise) the original circle from person \( J(n - 1, 3) \).
2. If the \( k^{th} \) person around the circle of \( n - 1 \) people is the last one remaining, then in the game that starts with \( n \) people, after one person is eliminated the first person in the remaining circle of \( n - 1 \) is person 4. The \( k^{th} \) person in this circle, is the \( (k + 3)^{th} \) person in the original circle.
3. For all \( n \geq 2 \), person \( J(n, 4) \) is four more around (clockwise) the original circle from person \( J(n - 1, 4) \).

Exercise 4. Here is the completed table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J(n) )</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>15</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
<th>25</th>
<th>26</th>
<th>27</th>
<th>28</th>
<th>29</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J(n) )</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>19</td>
<td>21</td>
<td>23</td>
<td>25</td>
<td>27</td>
<td>29</td>
</tr>
</tbody>
</table>

1. For all \( n \geq 2 \), person \( J(n) \) is two more around (clockwise) the original circle from person \( J(n - 1) \).
2. The next value of \( n \) for which \( J(n) = 1 \) will be \( n = 32 \). It appears that \( J(n) = 1 \) if and only if \( n \) is a power of 2.
3. Given \( n \) people originally, let \( m \) be the smallest power of 2 less than or equal to \( n \). Eliminate people \( 2, 4, \ldots, 2(n - m) \). This leaves the game with \( m \) people, the first of whom is person \( 2(n - m) + 1 \). According to the observation in part (b) of this exercise, this person will be the last person left at the end of the entire process.

**Exercise 5.** Here is the completed table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J(n) )</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>15</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>15</td>
<td>17</td>
</tr>
<tr>
<td>( F(n) )</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>19</td>
<td>21</td>
<td>23</td>
<td>1</td>
</tr>
</tbody>
</table>

1. For all \( n \geq 2 \), person \( F(n) \) is two more around (clockwise) the original circle from person \( F(n - 1) \).

2. The next value of \( n \) for which \( F(n) = 1 \) will be \( n = 48 \). It appears that \( F(n) = 1 \) if and only if \( n = 3 \cdot 2^k \) for some value of \( k \geq 0 \).

3. \( J(n) - F(n) = 2^k \) when the integer \( k \) can be chosen so that \( 3 \cdot 2^{k-1} \leq n < 2^{k+1} \), and \( F(n) - J(n) = 2^k \) when the integer \( k \) can be chosen so that \( 2^{k+1} \leq n < 3 \cdot 2^k \).

**Exercise 6.** Here is the completed table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
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<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(n) )</td>
<td>1</td>
<td>9</td>
<td>5</td>
<td>13</td>
<td>7</td>
<td>15</td>
<td>7</td>
<td>15</td>
<td>5</td>
<td>13</td>
<td>1</td>
<td>9</td>
<td>17</td>
<td>2</td>
<td>10</td>
</tr>
</tbody>
</table>

1. This answer will depend on the number of people in your class. Suppose there are 32 people in your class. Using the pattern of “adding 8” relative to the number in the circle, we find that \( J(32, 8) = 17 \).

2. Using a skip number of 60 will work for sure (see the next answer), but the smallest number that will work is \( k = 3 \).

3. For the game with \( n \) people, using \( k \) that is the least common multiple of the numbers in \( \{1, 2, 3, \ldots, n\} \) is guaranteed to work, but there are typically much smaller values.

Doug Ensley and Jim Hambin, Shippensburg University, from *Resources for Teaching Discrete Mathematics*, editor Hopkins, Mathematical Association of America 2009, pp45–53.
Session 11: Trains—Fibonacci via Pascal

Incredibly, “shallow diagonal” sums in Pascal’s triangle give Fibonacci numbers. For example,

\[
\binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 1 + 4 + 3 = 8 = F_6
\]

Our train interpretations give us a way to explain this. Remember that \(\binom{n}{k}\) counts length \(n + 1\) trains with \(k + 1\) cars. We use the interpretation of \(F_6\) as counting the length 7 trains with no length 1 cars.

More generally, we want to understand why

\[
\binom{n}{0} + \binom{n-1}{1} + \cdots = F_{n+1}
\]

We use the no 1s manifestation, showing \(T(n, 1)+T(n-1, 2)+\cdots = T^*_1(n+1)\). The trains on the left hand side are too short in various ways, but have the correct number of cars. Notice the the trains with \(k\) cars are exactly \(k\) too short! The correspondence comes by extending each car by 1.

It is instructional to describe the correspondence for each of the other two Fibonacci train interpretations. Benjamin & Quinn use the just red & white interpretation in *Proofs That Really Count*, MAA 2003. They go on to prove many Fibonacci identities from this interpretation; it would be interesting to see if either of the other interpretations makes the proofs even clearer.
Session 12: Connecting Take Away and Fibonacci

The game: Start with \( n \) tokens (with \( n > 2 \)). The winner is the player who takes the last token. The first player takes from 1 to \( n - 1 \) tokens. In subsequent turns, a player can take up to twice the number of tokens as her opponent in the previous turn. (For example, say Player 1 starts by taking 2 tokens. Player 2 may take 1, 2, 3, or 4 tokens; suppose she takes 1. Then Player 1 can take 1 or 2 tokens, etc.)

Which player has a winning strategy, and what is the strategy? The answer depends on \( n \) in an interesting way. If Player 1 takes \( n/3 \) or more tokens, Player 2 can win immediately.

\[
\begin{array}{c|c}
\text{n} & \text{winner} \\
2 & \text{Player 2: sequence of play is 1, 1} \\
3 & \text{Player 2: 1, 2 or 2, 1} \\
4 & \text{Player 1: 1, 1, 2 or 1, 2, 1} \\
5 & \text{Player 2: 1, 1, 1, 2 or 1, 1, 1, 2} \\
6 & \text{Player 1: 1, 2, 3 or 1, 1, 1, 2 or 1, 1, 2, 1} \\
7 & \text{Player 1: 2, 4, 1; 2, 3, 3; 2, 2, 3; 2, 1, 1, 2 or 2, 1, 1, 2} \\
\end{array}
\]

Listing all possible sequences of play becomes cumbersome, but the pattern is that Player 1 can win for any \( n \) that is not a Fibonacci number. The strategy depends on a theorem by Zeckendorf reported in 1952.

**Theorem 2.2.** Every integer can be represented uniquely as the sum of non-consecutive Fibonacci numbers.

To find the representation \( n \), use a “greedy” approach: use the greatest Fibonacci number less than or equal to \( n \), then repeat with the remaining difference until you reach 0. That this never gives consecutive Fibonacci numbers follows from \( F_n = F_{n-1} + F_{n-2} \). Proving that the representation is unique takes a little work.

Here are some Zeckendorf representations.

\[
\begin{array}{c|c|c|c}
1 &= 1 & 12 &= 8 + 3 + 1 & 23 &= 21 + 2 \\
2 &= 2 & 13 &= 13 & 24 &= 21 + 3 \\
3 &= 2 & 14 &= 13 + 1 & 25 &= 21 + 3 + 1 \\
4 &= 3 + 1 & 15 &= 13 + 2 & 26 &= 21 + 5 \\
5 &= 5 & 16 &= 13 + 3 & 27 &= 21 + 5 + 1 \\
6 &= 5 + 1 & 17 &= 13 + 3 + 1 & 28 &= 21 + 5 + 2 \\
7 &= 5 + 2 & 18 &= 13 + 5 & 29 &= 21 + 8 \\
8 &= 8 & 19 &= 13 + 5 + 1 & 30 &= 21 + 8 + 1 \\
9 &= 8 + 1 & 20 &= 13 + 5 + 2 & 31 &= 21 + 8 + 2 \\
10 &= 8 + 2 & 21 &= 21 & 32 &= 21 + 8 + 3 \\
11 &= 8 + 3 & 22 &= 21 + 1 & 33 &= 21 + 8 + 3 + 1 \\
\end{array}
\]

Strategy for the game (developed by Whinihan, 1963): If you can take the entire pile, do so and win. If you cannot and you are facing \( n \) tokens, \( n \) not a Fibonacci number, write its Zeckendorf representation and remove the smallest
term. (If you are presented a Fibonacci number, and your opponent’s previous
move does not allow you to finish the game, take 1 and hope for an error.) It’s
worth thinking about through the 17 token game: can you start with 4 tokens,
or does it need to be 1?

Postscript: All of these take-away games are variants of a combinatorial
game know as Nim. Many variants lead to accessible interesting mathematics
appropriate for classroom use, although you might want to use a different name
in class to avoid students searching the web for solutions.