There are 7 problems on this exam. The best 5 will be taken for the final score.

1. Let \( f \) be a real valued function defined on the real numbers \( \mathbb{R} \).
   (i) Give the definition for \( f \) to be uniformly continuous on \( \mathbb{R} \).  
   (ii) Prove that: if, for some constant \( M > 0 \), \( f \) satisfies \[ |f(x) - f(y)| \leq M|x - y| \] for all \( x, y \in \mathbb{R} \), then \( f \) is uniformly continuous on \( \mathbb{R} \).

2. Give an example of a real valued function \( f \) and a set \( S \subseteq \mathbb{R} \) such that \( f \) is continuous on \( S \) but not uniformly continuous on \( S \). Prove all your claims.

3. Let \( f(x) = \frac{g(x) - \cos x}{x} \) if \( x \neq 0 \) and \( f(x) = a \) if \( x = 0 \), where \( g''(x) \) exists and is continuous for all \( x \), and where \( g(0) = 1 \), \( g'(0) = 2 \), and \( g''(0) = 4 \).
   (i) Make the value \( a \) such that \( f(x) \) is continuous at \( x = 0 \).
   (ii) Establish the value \( f'(0) \) directly; do not assume continuity of \( f'(x) \) at \( x = 0 \).
   (iii) Show that \( \lim_{x \to 0} f'(x) \) exists and equals the value \( f'(0) \) found in part (ii).

4. In each case determine whether the given sequence of functions \( f_n(x) \) converges uniformly or not. Justify your answers.
   (i) \( f_n(x) = \frac{\sin x}{\sqrt{n}} \) on \( \mathbb{R} \).  
   (ii) \( f_n(x) = \sum_{k=1}^{n} x^k \) on \([-1/2, 1/2]\).

5. In each case determine whether the given series of numbers converges or diverges. Justify your answers.
   (i) \( \sum_{n=2}^{\infty} \frac{\ln n}{n^2} \).  
   (ii) \( \sum_{n=1}^{\infty} \frac{(-1)^n n^3}{(1.1)^n} \).

6. Assume that the sequence of real valued continuous functions \( f_n \) converges uniformly to \( f \) on \([0, 1]\).
   (i) Prove that \( f \) is continuous.
   (ii) Prove that \[ \lim_{n \to \infty} \int_{0}^{1} f_n \, dx = \int_{0}^{1} f \, dx. \]

7. (i) Let \( A = \{(x, x^2) \in \mathbb{R}^2 : 0 \leq x \leq 1\} \). Show that \( A \) is closed in \( \mathbb{R}^2 \) equipped with the Euclidean distance.
   (ii) Let \( A_1 \) be the subset of \( A \) as follows: \( A_1 = \{(x, x^2) \in A : x \text{ is rational}\} \). Show that \( A_1 \) is not closed in \( \mathbb{R}^2 \).
1. \( f: \mathbb{R} \to \mathbb{R} \) is said to be uniformly continuous if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for any \( x_1, x_2 \in \mathbb{R} \) with \( |x_1 - x_2| < \delta \) we have \( |f(x_1) - f(x_2)| < \varepsilon \).

(ii) Assume \( |f(x) - f(y)| \leq M |x - y| \) all \( x, y \in \mathbb{R} \) for some constant \( M > 0 \). Then, given \( \varepsilon > 0 \), choose \( \delta = \varepsilon / M \). We claim that this choice of \( \delta \) satisfies the definition in (i). Indeed, by (ii), if \( |x_1 - x_2| < \delta \) then \( |f(x_1) - f(x_2)| \leq M |x_1 - x_2| \leq M \delta = M \frac{\varepsilon}{M} = \varepsilon \).

Hence (i) holds.

2. We find \( f: S \to \mathbb{R} \) such that \( f \) is continuous on \( S \) but not uniformly continuous on \( S \).

Indeed, we choose \( S = (0, 1] \) and \( f(x) = \sqrt{x} \).

(a) We know that \( f \) is continuous on \( S \) because \( f \) is a rational function with no zeroes in the denominator on \( S \).

(b) We show that \( f \) is not uniformly continuous by finding \( \varepsilon > 0 \) and sequences \((x_n)\) and \((y_n)\) in \( S \) with \( |x_n - y_n| \to 0 \) but \( |f(x_n) - f(y_n)| \geq \varepsilon \) for all \( n \).

Indeed, we take \( x_n = 1/n \) and \( y_n = \frac{1}{n+1} \) and \( \varepsilon_0 = 1 \).

We have \( |f(x_n) - f(y_n)| = |\sqrt{x_n} - \sqrt{y_n}| = |n - (n+1)| = 1 \geq \varepsilon_0 \).

3. \( f(x) = \begin{cases} \frac{g(x) - \cos x}{x}, & x \neq 0 \\ a & x = 0 \end{cases} \)

\( g(x) \) continuous all \( x \)
\( g(0) = 1, g'(0) = 2, g''(0) = 4 \).

(i) \( \lim_{x \to 0} f(x) = a \iff \lim_{x \to 0} g(x) - \cos x = a \)

But \( g(x) - \cos x \) is differentiable with \( \lim g(x) - \cos x = g(0) - \cos(0) = 1 - 1 = 0 \)

So L'Hôpital's rule applies and we obtain \( \lim_{x \to 0} g'(x) - \sin x = 2 - 0 = 2 \).
June 2003 Comp. Exam Analysis

3. (Continued) Therefore we take \( a = 2 \) to make \( f \) continuous at \( x = 0 \).

(i) \( f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} 
= \lim_{x \to 0} \frac{g(x) - \cos x}{x} - 2 \lim_{x \to 0} \frac{g(x) - \cos x - 2x}{x} 
= \lim_{x \to 0} \frac{g'(x) + \sin x - 2}{2x} 
= \lim_{x \to 0} \frac{g''(x) + \cos x}{2} 
= \frac{4 + 1}{2} = 2.5 \)

(ii) \( f'(x) = \frac{d}{dx} g(x) - \cos x 
= (g'(x) + \sin x) - (g(x) - \cos x) \), \( x \neq 0 \).

So \( \lim_{x \to 0} f'(x) = \lim_{x \to 0} x g'(x) - g(x) + x \sin x + \cos x 
= \lim_{x \to 0} x g''(x) + \cos x 
= \lim_{x \to 0} \frac{g''(x) + \cos x}{2} 
= \frac{4 + 1}{2} = 2.5 \).

4. Determine uniform convergence of the sequence

(i) \( f_n(x) = \frac{\sin x}{\sqrt{n}} \) on \( \mathbb{R} \). \( f_n \) converges uniformly to \( f \equiv 0 \) on \( \mathbb{R} \).

Since \( \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \sup_{x \in \mathbb{R}} \left| \frac{\sin x}{\sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n}} \to 0 \) as \( n \to \infty \).
(3)

June, 2008, exam in analysis.

4.(iii) \( f_n(x) = \sum_{k=1}^{\infty} x^k \) on \([-\frac{1}{2}, \frac{1}{2}]\).

This sequence of functions converges uniformly means that \( \sum_{k=1}^{\infty} x^k \) converges uniformly. We apply the Weierstrass M-test as follows. \( M_k = \sup_{x \in [-\frac{1}{2}, \frac{1}{2}]} |x^k| = (\frac{1}{2})^k \). We show that \( \sum_{k=1}^{\infty} M_k \) is a convergent series of numbers. Indeed, by the ratio test

\[
\lim_{k \to \infty} \frac{M_{k+1}}{M_k} = \lim_{k \to \infty} \frac{(\frac{1}{2})^{k+1}}{(\frac{1}{2})^k} = \frac{1}{2} < 1
\]

Therefore, by the ratio test, \( \sum_{k=1}^{\infty} M_k \) converges. Hence \( \sum_{k=1}^{\infty} x^k \) converges uniformly on \([-\frac{1}{2}, \frac{1}{2}]\). The limit is continuous since \( f_n(x) \) is continuous for each \( n \). In fact

\[
\lim_{n \to \infty} f_n(x) = \frac{x}{1-x} \sin(\pi + x + x^2 + \cdots) = \frac{x}{1-x^2}, \quad x \in (-1, 1).
\]

Note that the convergence is not uniform on \( S = (-1, 1) \) because

\[
\sup_{x \in (-1, 1)} |f_n(x) - f(x)| = \sup_{x \in (-1, 1)} \sum_{k=n+1}^{\infty} x^k
\]

for every \( N \).

Moreover, \( \sup_{x \in (-1, 1)} |f_n(x) - f(x)| = \infty \) as \( n \to \infty \).


(i) \[ \sum_{n=2}^{\infty} \frac{\ln n}{n^2} \]

Apply integral test with \( f(x) = \frac{\ln x}{x^2} \).

\[
\int_{2}^{\infty} \frac{\ln x}{x^2} \, dx = -\frac{\ln x}{x^2} \bigg|_{2}^{\infty} = \frac{\ln 2}{2}
\]

for \( u = \ln x, \quad du = \frac{1}{x} \, dx \), \( x = e^u, \quad \frac{1}{x} = e^{-u} \).
5(i) Now \( \int_0^\infty \frac{1}{e^{e^u}} du = -e^{-e^u} \bigg|_0^\infty = -e^0 + e^{-\ln 2} = 0 + \frac{1}{\ln 2} \). So, the series converges by the integral test.

Alternatively, use limit comparison with \( \sum_{n=2}^\infty \frac{1}{n^{3/2}} \).

We have \( \lim_{n \to \infty} \frac{\ln n}{n^{1/2}} \).

\[
\lim_{n \to \infty} \frac{\ln n}{n^{1/2}} = \lim_{n \to \infty} \frac{\ln 2}{n^{1/2}} = \lim_{n \to \infty} 2n^{1/2} = 0 \quad \text{since} \quad \sum_{n=2}^\infty \frac{1}{n^{3/2}} \text{ converges.}
\]

Therefore \( \sum_{n=2}^\infty \frac{\ln n}{n^2} \) converges since \( \sum_{n=2}^\infty \frac{1}{n^{3/2}} \) converges; the latter series converges by the p-series test with \( p = \frac{3}{2} > 1 \).

(ii) \( \sum_{n=1}^\infty (-1)^n \frac{n^3}{(1,1)^n} \) is an alternating series.

We will show that \( a_n = \frac{n^3}{(1,1)^n} \) converges monotonically to \( 0 \). This follows if the ratio \( \frac{a_{n+1}}{a_n} \) has a limit less than 1. But

\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{(1,1)^{n+1}} / \frac{n^3}{(1,1)^n} \to \frac{1^3}{1} = \frac{1}{(1,1)} < 1.
\]

So \( \lim a_n = 0 \) in particular and also \( a_n \) is eventually monotone. Therefore \( \sum_{n=1}^\infty a_n \) converges by the alternating series test.

6. Assume \( f_n \) converges uniformly to \( f \) on \([0,1]\) where \( f_n \) is continuous for each \( n \). Then \( f \) is continuous as follows. (more)
6. Let \( \varepsilon > 0 \) and let \( x \in [0, 1] \). Find \( N \) so large that \( \sup_{t \in [0, 1]} |f_N(t) - f(t)| < \varepsilon / 3 \). (by uniform convergence)

Then, find \( \delta > 0 \) such that \( |f_N(y) - f_N(x)| < \varepsilon / 3 \) if \( |x - y| < \delta \) (by continuity of \( f_N(t) \) at \( t = x \)).

Thus, for \( |x - y| < \delta \):

\[
|f(y) - f(x)| = |(f(y) - f_N(y)) + (f_N(y) - f_N(x)) + (f_N(x) - f(x))| \leq \varepsilon / 3 + \varepsilon / 3 + \varepsilon / 3 \quad \text{by the triangle inequality}
\]

and the above condition.

6 (ii) \[
\left| \int_0^1 f_n(x) \, dx - \int_0^1 f(x) \, dx \right| = \left| \int_0^1 (f_n(x) - f(x)) \, dx \right| \leq \int_0^1 |f_n(x) - f(x)| \, dx \leq \int_0^1 \sup_{x \in [0, 1]} |f_n(x) - f(x)| \, dx
\]

\[
= \int_0^1 \delta_n \, dx = \delta_n (1 - 0) = \delta_n \to 0
\]

where \( \delta_n = \sup_{x \in [0, 1]} |f_n(x) - f(x)| \to 0 \) by definition of uniform convergence.

7. (i) \( A = \{ (x, x^2) \in \mathbb{R}^2 | 0 \leq x \leq 1 \} \). \( A \) is closed since \( A \) is the inverse image of a closed set \( C \subseteq \mathbb{R} \) by the continuous mapping \( F(x, y) = y-x^2 : [0, 1] \times \mathbb{R} \to \mathbb{R} \).

Indeed \( F \) is continuous as a composition and difference of elementary continuous functions. Claim \( A = F^{-1}([0]) \), that is \( C \) is the singleton set \( \{0\} \subseteq \mathbb{R} \) and so \( C \) is trivially closed.

Now \( F^{-1}([0]) = \{(x, y) | y-x^2 = 0\} = \{(x, x^2) | x \in [0, 1]\} \).
7(ii) \( A_1 = \{ (x, x^2) \mid x \text{ is rational in } [0, 1] \} \).

We show \( A_1 \) is not closed in \( \mathbb{R}^2 \) by finding a limit point of \( A_1 \) that does not belong to \( A_1 \).

Put \( x_n = \text{rational approximation to } \sqrt{2}/2 \), so that
\[ x_n \to \frac{\sqrt{2}}{2} \] (possible by density of rationals in the reals).

Then \( (x_n, x_n^2) \to \left( \frac{\sqrt{2}}{2}, \frac{1}{2} \right) \notin A_1 \), since \( \frac{\sqrt{2}}{2} \in \mathbb{R} \).

\[ \text{instead,} \]