1. Determine if each of the following statements is TRUE or FALSE by giving a short proof or a counterexample.

(a) If \( A \) is a nonsingular square matrix which is diagonalizable, then so is \( A^{-1} \).

(b) If a linear operator \( T \) on a real inner product space \( V \) satisfies \( <Tv, v> = 0 \) for all \( v \in V \), then \( T \) must be the zero transformation.

(c) The matrices \( \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \) and \( \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \) are similar.

2. For a fixed \( a \in \mathbb{R} \), consider the subspace \( W = \{ f \in P_n(\mathbb{R}) | f(a) = f'(a) = 0 \} \) of \( P_n(\mathbb{R}) \), the space of real polynomials with degree at most \( n \). Determine the dimension of \( W \) and write a basis for \( W \).

3. Given a linear operator \( T \) on a finite dimensional vector space \( V \), satisfying \( T^2 = T \).

(a) Using the dimension theorem, show that \( N(T) \oplus R(T) = V \).

(b) Identify all eigenvalues of \( T \) and the corresponding eigenspaces and show that \( T \) is diagonalizable.

4. (i) If \( A \) and \( Q \) are unitary matrices, show that \( U = Q^{-1}AQ \) is also unitary.

(ii) If a unitary matrix \( U \) is also upper triangular, show that \( U \) must be diagonal.

(iii) State Schur’s theorem and apply it together with parts (i) and (ii) to show directly that any unitary operator \( T \) on a finite dimensional complex inner product space is diagonalizable.

(iv) Does the conclusion in part (iii) remain valid if \( V \) is a real inner product space? Justify your answer.

5. Given the \( 4 \times 4 \) matrix \( A = \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \),

(a) Find the Jordan canonical form \( J \) of the matrix \( A \) [You do NOT need to compute generalized eigenvectors in this part].

(b) Determine the minimal polynomial of \( A \).

(c) Show that \( A \) is nonsingular and express \( A^{-1} \) as a polynomial (of least degree) of \( A \).

(d) Find an invertible matrix \( Q \) such that \( A = QJQ^{-1} \).