1. In $\mathbb{R}^2$, $N$ is the usual natural basis $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$; let $B$ be the basis $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. $T$ is a linear transformation on $\mathbb{R}^2$ defined by its action on $N$: $T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$. $T^2 = T \circ T$.

a) Find the matrix (often denoted $[T^2]_N$) that represents $T^2$ with respect to the basis $N$.

b) Find the matrix (often denoted $[T^2]_B$) that represents $T^2$ with respect to the basis $B$.

c) Find a basis $C$ and a diagonal matrix $D$ such that the matrix that represents $T$ with respect to $C$ (i.e., $c(T|C)$) is $D$. (Note that this question refers to $T$, not $T^2$.)

2. Let $\mathcal{V}$ be a vector space and let $\{v_1, v_2, \ldots, v_k\}$ be a linearly independent set of vectors in $\mathcal{V}$. If $T$ is a non-singular linear transformation on $\mathcal{V}$, show that $\{Tv_1, Tv_2, \ldots, Tv_k\}$ is also a linearly independent set. (Clearly indicate the step in your proof where you use the hypothesis that $T$ is non-singular.)

3. Let $\mathcal{P}_n$ be the vector space over $\mathbb{R}$ of all polynomials of degree $n$ or less ($n \geq 3$) with real coefficients. Let $f$ and $g \in \mathcal{P}_n$ be defined as $f(x) = x + 1$ and $g(x) = x - 1$.

a) Prove: $f$ and $g$ are linearly independent in $\mathcal{P}_n$.

b) Let $S$ be the subspace of $\mathcal{P}_n$ spanned by $f$ and $g$. Define a linear transformation $T$ on $\mathcal{P}_n$ such that the kernel (or null space) of $T$ is $S$.

4. Let $A = \begin{pmatrix} 1 & -2 & 3 & 0 \\ -2 & 4 & -1 & 1 \\ 3 & -6 & 4 & 1 \end{pmatrix}$.

a) Find a basis for the row space of $A$.

b) Find a basis for the column space of $A$.

c) Find a basis for the null space of $A$.

d) What is the dimension of the null space of $A^T$?

5. Let $A = \begin{pmatrix} 6 & 4 & 10 \\ 8 & 0 & 5 \\ 0 & 0 & 3 \\ 9 & 6 & 7 \end{pmatrix}$, but note that two entries of the last column are missing. The characteristic polynomial of $A$ is known to be $p_A(t) = t^4 - 12t^3 - 5t^2 + 96t$. What are the missing entries?

6. Let $\mathcal{C}^n$ be the vector space of $n$-tuples over $C$, the field of complex numbers, and let $\langle \cdot, \cdot \rangle$ be the usual inner product on $\mathcal{C}^n$. Let $\|\cdot\|$ be the vector norm induced by this inner product. I.e., $\|v\| = \langle v, v \rangle^{1/2}$. Let $U$ be an $n \times n$ unitary matrix. Prove that for every $v \in \mathcal{C}^n$, $\|Uv\| = \|v\|$.

7. Let $A$, an $n \times n$ matrix with entries from $C$, the field of complex numbers, have the property that $A^3 = I_n$, where $I_n$ is the $n \times n$ identity matrix.

a) What are the possible eigenvalues of $A$?

b) Prove: $A$ is diagonalizable.

c) Suppose further that $n = 3$ and all entries of $A$ are real numbers. List all possible non-similar Jordan Canonical Forms for $A$. 