Master of Science Exam in Applied Mathematics
Analysis − August 19, 2005

There are 10 problems here. The best 7 will be used for the grade.

1. Consider the space of functions
\[ C = \{ f : f \text{ maps } [0, 1] \to \mathbb{R} \text{ and } f \text{ is continuous} \} \]
and define
\[ d(f, g) := \sup \{|f(x) - g(x)| : x \in [0, 1]\}. \]

(a) Show that \( d(f, g) \) is a metric on \( C \).
(b) Let
\[ F := \{ f \in C : 0 \leq f(x) \leq 1 \text{ for } x \in [0, 1]\}. \]
Show that \( F \) is (i) bounded and (ii) closed as a set in the metric space \( C \) under the metric \( d(f, g) \).
(c) Define a sequence of functions \( \{f_n\} \) in \( C \) by
\[ f_n(x) = x^n, \ x \in [0, 1]. \]
Show that there is no subsequence \( \{f_{n_k}\} \) of the given sequence that converges in \( (C, d) \).

SOLUTION
(a) There are 4 properties to verify
1) \( d(f, f) = 0 \); this is clear.
2) \( d(f, g) = 0 \Rightarrow f(x) = g(x), \ \text{all } x. \)
   Proof: If not, there is a \( t \) for which \( f(t) \neq g(t) \). Then
   \[ d(f, g) \geq |f(t) - g(t)| > 0 \]
3) \( d(f, g) = d(g, f) \); this is clear because \( |f(x) - g(x)| = |g(x) - f(x)|, \ \text{all } x. \)
4) The triangle inequality:
   \[ d(f, g) \leq d(f, h) + d(h, g). \]
   Proof: For all \( x, \)
   \[ |f(x) - g(x)| = |(f(x) - h(x)) + (h(x) - g(x))| \leq |f(x) - h(x)| + |h(x) - g(x)| \leq d(f, h) + d(h, g). \]
   Hence
   \[ d(f, g) = \sup \{|f(x) - g(x)| : x \in [0, 1]\} \leq d(f, h) + d(h, g). \]

b) (i) \( F \) is bounded because
   \[ d(f, g) \leq d(f, 0) + d(0, g) \leq 2. \]
(ii) \( F \) is closed because if \( f_n \in F \) and
   \[ \lim_{n \to \infty} d(f_n, g) = 0 \Rightarrow \lim_{n \to \infty} (f_n(x) - g(x)) = 0, \ \text{all } x \Rightarrow \lim_{n \to \infty} f_n(x) = g(x) \Rightarrow 0 \leq g(x) \leq 1. \]
c) If there is a subsequence \( f_{n_k} \) in \( \mathcal{F} \) and a function \( g \in \mathcal{F} \) for which
\[
\lim_{k \to \infty} d(f_{n_k}, g) = 0
\]
then for all \( x \)
\[
\lim_{k \to \infty} |x^{n_k} - g(x)| = 0; \quad \lim_{k \to \infty} x^{n_k} = g(x).
\]
However
\[
\lim_{k \to \infty} x^{n_k} = \begin{cases} 
0 & \text{if } x \in [0, 1) \\
1 & \text{if } x = 1
\end{cases}
\]
and this function is not continuous on \([0, 1]\).

2. Define a sequence \( \{a_n\} \) in \([-1, 1] \subset \mathbb{R} \) by \( a_n = \sin(n) \). Even though there seems to be no apparent pattern in the values of this sequence, it must have a convergent subsequence.
State the relevant theory that proves the existence of such a convergent subsequence.

**SOLUTION** The interval \([-1, 1]\) is a compact subset of \( \mathbb{R} \) and one property of compactness of a set in \( \mathbb{R} \) is that every sequence in the set has a convergent subsequence.

3. Define \( g_n : [0, 1] \to \mathbb{R} \) by \( g_n(x) = e^{-nx} \).
   (a) Show that \( \lim_{n \to \infty} g_n(x) = 0 \) uniformly on \([0, 1]\).
   (b) Prove in addition that \( \sum_{n=1}^{\infty} g_n(x) \) converges uniformly on \([0, 1]\).

**SOLUTION.**
(a) \( e^{-nx} \leq e^{-n}, \) all \( x \in [0, 1] \) and \( \lim_{n \to \infty} e^{-n} = 0. \)
(b) \( |g_n(x)| \leq e^{-n} \) for all \( x \in [0, 1] \) and the geometric series \( \sum_{n=1}^{\infty} e^{-n} \) converges, and so by

W-M test \( \sum_{n=1}^{\infty} g_n(x) \) converges uniformly.
4. Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( g(x,y) = \sin(x/3) + \cos(y/3) \).

   (a) Show that the gradient vector of partial derivatives \( \nabla g = (\partial g/\partial x, \partial g/\partial y) \) satisfies
   \[ \|\nabla g\| \leq 1/2 \] for all \((x,y) \in \mathbb{R}^2\), (the vector norm is the Euclidean norm).

   (b) The Mean Value Theorem asserts that if a function \( f : \mathbb{R}^2 \to \mathbb{R} \) has continuous partial
derivatives on all of \( \mathbb{R}^2 \) then for all \((x_0,y_0),(x,y) \in \mathbb{R}^2\) there exists \( \theta = \theta(x_0,y_0,x,y) \in (0,1) \) such that
   \[ f(x,y) = f(x_0,y_0) + (\nabla f) \cdot (x - x_0, y - y_0), \]
   where the dot product is indicated in the formula and where each partial derivative in the
   gradient vector \( \nabla f = (\partial f/\partial x, \partial f/\partial y) \) is evaluated at the point
   \((x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)) \in \mathbb{R}^2\). Conclude that
   \[ |g(x,y) - g(x_0,y_0)| \leq (1/2)\|(x - x_0, y - y_0)\| \]
   for all \((x_0,y_0),(x,y) \in \mathbb{R}^2\).

   (c) Define also \( h : \mathbb{R}^2 \to \mathbb{R} \) by \( h(x,y) = \sin(x/5) + \cos(y/5) \), and define the mapping
   \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( F(x,y) = (g(x,y), h(x,y)) \). Let \((a_0,b_0) = (0,0) \in \mathbb{R}^2\) and inductively define
   \((a_{n+1},b_{n+1}) = F(a_n,b_n) \in \mathbb{R}^2\). Verify that the mapping \( F \) on \( \mathbb{R}^2 \) is indeed a contraction and so
   conclude by the Contraction Mapping Theorem (check the hypotheses please) that the
   sequence \( \{(a_n,b_n)\} \) has a limit in \( \mathbb{R}^2 \).
5. Let \( f : (0, 1] \to \mathbb{R} \)
   (a) Define uniform continuity for \( f \) on \( (0, 1] \)
   (b) Assume \( f \) is uniformly continuous. Let \( \{x_n\} \) be a Cauchy sequence in \( (0, 1] \). Show that \( \{f(x_n)\} \) is a Cauchy sequence in \( \mathbb{R} \).

   **SOLUTION**
   (a) For any \( \varepsilon > 0 \) there is a \( \delta > 0 \) for which
   \[
   x \in (0, 1] \text{ and } t \in (0, 1] \text{ and } |x - t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon.
   \]
   (b) There is an \( N \) for which
   \[
   n \geq m \geq N \Rightarrow |x_n - x_m| < \delta.
   \]

   Hence
   \[
   |f(x_n) - f(x_m)| < \varepsilon.
   \]

6. Define \( f \) on \( \mathbb{R} \) by
   \[
   f(x) = \begin{cases} 
   \frac{\sin x}{x} & \text{if } x \neq 0 \\
   1 & \text{if } x = 0
   \end{cases}
   \]

   (a) Show that \( f \) is continuous on \( \mathbb{R} \).
   (b) Show that \( f'(0) \) exists and find \( f'(0) \).

   **Hint:** One approach is l’Hospital’s rule.

   **SOLUTION.**
   (a). Using l'Hospital's rule,
   \[
   \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{(\sin x)'}{(x)'} = \lim_{x \to 0} \frac{\cos x}{1} = \cos(0) = 1.
   \]

   This shows that \( f \) is continuous at 0; for \( x \neq 0 \), continuity follows from the continuity of \( \sin(x) \) and of \( x \).

   (b) The difference quotient for \( f'(0) \) is
   \[
   \frac{\sin x - 1}{x} = \frac{\sin(x) - x}{x^2}
   \]

   and using l'Hospital’s rule twice:
   \[
   \lim_{x \to 0} \frac{\sin x - 1}{x} = \lim_{x \to 0} \frac{\sin(x) - x}{x^2} = \lim_{x \to 0} \frac{\cos(x) - 1}{2x} = \lim_{x \to 0} \frac{-\sin(x)}{2} = 0.
   \]

   Thus \( f'(0) \) exists and
   \[
   f'(0) = 0.
   \]

   The \( x \neq 0 f(x) \) is the quotient of functions having a derivative and so has a derivative.
7. Let $K$ be a compact set in a metric space $(X, d)$ and let $f$ be a continuous real valued function on $(X, d)$.
   (a) Prove that there is an $x \in K$ for which
   \[
   f(x) = \sup \{ f(t) : t \in K \}.
   \]
   (b) Give an example of a set $K \subset \mathbb{R}$ and a function $f$ on $K$ for which the assertion fails.

   **SOLUTION** (a) There is a sequence $t_k$ in $K$ for which
   \[
   \lim_{k \to 0} f(t_k) = \sup \{ f(t) : t \in K \}.
   \]
   Since $K$ is compact there is a subsequence $t_{k_n}$ which converges, say
   \[
   x = \lim_{n \to \infty} t_{k_n}.
   \]
   By continuity of $f$ we have
   \[
   f(x) = \lim_{n \to \infty} f(t_{k_n}) = \lim_{k \to 0} f(t_k) = \sup \{ f(t) : t \in K \}.
   \]
   (b) Let $K = \mathbb{R}$. Let
   \[
   f(x) = 1 - \frac{1}{1 + |x|}.
   \]
   Then $f(x) < 1$ for all $x \in \mathbb{R}$ but
   \[
   \sup \{ f(t) : t \in K \} = 1.
   \]

8. One form of completeness of the real numbers $\mathbb{R}$ is that every bounded increasing sequence converges. Use this property to prove that every Cauchy sequence in $\mathbb{R}$ converges.

   **SOLUTION.** Let $(x_n)$ be a Cauchy sequence. First, there is a positive integer $N$ for which
   \[
   n \geq N \Rightarrow |x_n - x_N| < 1 \Rightarrow |x_n| < 1 + |x_N|.
   \]
   Hence the sequence $(x_n)$ is bounded. Define
   \[
   y_k = \sup \{ x_n : n \leq k \}.
   \]
   The sequence $(y_k)$ is increasing and so $z = \lim_{k \to \infty} y_k$ exists. For $\varepsilon > 0$, there is a $K$ for which
   \[
   k \geq K \Rightarrow |y_k - z| < \frac{\varepsilon}{2}.
   \]
   Since $(x_n)$ is a Cauchy sequence, there is an $N \geq K$ for which
   \[
   n, k \geq N \Rightarrow |x_n - y_k| < \frac{\varepsilon}{2}.
   \]
   Hence
   \[
   n \geq N \Rightarrow |x_n - z| < |x_n - y_k| + |y_k - z| < \varepsilon
   \]
   showing that
   \[
   \lim_{n \to \infty} x_n = z.
   \]
9. Find the radius of convergence of each power series \( \sum_{n=0}^{\infty} a_n x^n \).

\[
\begin{align*}
\text{a) } a_n &= n & \text{b) } a_n &= \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{n} & \text{if } n > 0 \end{cases} & \text{c) } a_n &= \begin{cases} 1 & \text{if } n = 2^k \text{, some } k \geq 0 \\ 0 & \text{otherwise} \end{cases}
\end{align*}
\]

**SOLUTION**
(a) \( r = 1 \) by ratio test.
(b) \( r = 1 \) by ration test.
(c) \( r = 1 \) by root test:
\[
r = \limsup_{n \to \infty} \left( a_n \right)^{\frac{1}{n}} = 1.
\]

10. Let \( f \) be a function with domain \( D \subset \mathbb{R}^2 \), range \( \mathbb{R}^2 \), and defined by
\[
f(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}.
\]

a) What is the natural domain \( D \) of \( f \)?
b) The local inverse mapping theorem applies to \( f \). Find the set \( J \) for which the theorem guarantees a local inverse.

**SOLUTION**
(a) \( D = \{ (x, y) : x \neq 0 \text{ and } y \neq 0 \} \).
(b) Let
\[
f(x, y) = (u(x, y), v(x, y))
\]

The derivatives
\[
\frac{\partial u}{\partial x} = \frac{1}{y}, \quad \frac{\partial u}{\partial y} = -\frac{x}{y^2}
\]
\[
\frac{\partial v}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{x}
\]

for \( (x, y) \in D \) the Jacobian matrix is thus
\[
M(f)(x, y) = \begin{bmatrix}
\frac{1}{y} & -\frac{x}{y^2} \\
-\frac{y}{x^2} & \frac{1}{x}
\end{bmatrix}
\]

and the determinant is:
\[
\det[M(f)(x, y)] = \det\begin{bmatrix}
\frac{1}{y} & -\frac{x}{y^2} \\
-\frac{y}{x^2} & \frac{1}{x}
\end{bmatrix} = \frac{1}{xy} - \frac{x}{y^2} \cdot \frac{y}{x^2} = 0.
\]

Thus the local inverse mapping does not apply for every \( (x, y) \in D \). So the set is
\[
J = \emptyset.
\]