Master of Science Exam in Applied Mathematics
Analysis – August 19, 2005

There are 10 problems here. The best 7 will be used for the grade.

1. Consider the space of functions

\[ C = C([0,1], \mathbb{R}) = \{ f : f \text{ maps } [0,1] \to \mathbb{R} \text{ and } f \text{ is continuous} \} \]

and define

\[ d(f,g) := \sup\{|f(x) - g(x)| : x \in [0,1]\}. \]

(a) Show that \(d(f,g)\) is a metric on \(C\).

(b) Let

\[ \mathcal{F} := \{ f \in C : 0 \leq f(x) \leq 1 \text{ for } x \in [0,1]\}. \]

Show that \(\mathcal{F}\) is (i) bounded and (ii) closed as a set in the metric space \(C\) under the metric \(d(f,g)\).

(c) Define a sequence of functions \(\{f_n\}\) in \(C\) by

\[ f_n(x) = x^n, \quad x \in [0,1]. \]

Show that there is no subsequence \(\{f_{n_k}\}\) of the given sequence that converges in \((C,d)\).

2. Define a sequence \(\{a_n\}\) in \([0,1] \subset \mathbb{R}\) by \(a_n = \sin(n)\). Even though there seems to be no apparent pattern in the values of this sequence, it must have a convergent subsequence. State the relevant theory that proves the existence of such a convergent subsequence.

3. Define \(g_n : [0,1] \to \mathbb{R}\) by \(g_n(x) = e^{-nx^2}\).

   (a) Show that \(\lim_{n \to \infty} g_n(x) = 0\) uniformly on \([0,1]\).

   (b) Prove in addition that \(\sum_{n=1}^{\infty} g_n(x)\) converges uniformly on \([0,1]\).

4. Let \(g : \mathbb{R}^2 \to \mathbb{R}\) be defined by \(g(x,y) = \sin(x/3) + \cos(y/3)\).

   (a) Show that the gradient vector of partial derivatives \(\nabla g = (\partial g/\partial x, \partial g/\partial y)\) satisfies \(\|\nabla g\| \leq 1/2\) for all \((x,y) \in \mathbb{R}^2\), (the vector norm is the Euclidean norm).

   (b) The Mean Value Theorem asserts that if a function \(f : \mathbb{R}^2 \to \mathbb{R}\) has continuous partial derivatives on all of \(\mathbb{R}^2\) then for all \((x_0, y_0), (x,y) \in \mathbb{R}^2\) there exists \(\theta = \theta(x_0, y_0, x, y) \in (0,1)\) such that

\[ f(x,y) = f(x_0,y_0) + (\nabla f) \cdot (x-x_0, y-y_0), \]

where the dot product is indicated in the formula and where each partial derivative in the gradient vector \(\nabla f = (\partial f/\partial x, \partial f/\partial y)\) is evaluated at the point \((x_0 + \theta(x-x_0), y_0 + \theta(y-y_0)) \in \mathbb{R}^2\). Conclude that

\[ |g(x,y) - g(x_0,y_0)| \leq (1/2) ||(x-x_0, y-y_0)|| \]

for all \((x_0,y_0), (x,y) \in \mathbb{R}^2\).

   (c) Define also \(h : \mathbb{R}^2 \to \mathbb{R}\) by \(h(x,y) = \sin(x/5) + \cos(y/5)\), and define the mapping \(F : \mathbb{R}^2 \to \mathbb{R}^2\) by \(F(x,y) = (g(x,y), h(x,y))\). Let \((a_0, b_0) = (0,0) \in \mathbb{R}^2\) and inductively define \((a_{n+1}, b_{n+1}) = F(a_n, b_n) \in \mathbb{R}^2\). Verify that the mapping \(F\) on \(\mathbb{R}^2\) is indeed a contraction and so conclude by the Contraction Mapping Theorem (check the hypotheses please) that the sequence \(\{(a_n, b_n)\}\) has a limit in \(\mathbb{R}^2\).
5. Let $f : [0, 1] \to \mathbb{R}$
(a) Define uniform continuity for $f$ on $(0, 1]$.
(b) Assume $f$ is uniformly continuous. Let $\{x_n\}$ be a Cauchy sequence in $(0, 1]$. Show that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

6. Define $f$ on $\mathbb{R}$ by
$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$
(a) Show that $f$ is continuous on $\mathbb{R}$.
(b) Show that $f'(0)$ exists and find $f'(0)$.
Hint: One approach is l'Hôpital's rule.

7. Let $K$ be a compact set in a metric space $(X, d)$ and let $f$ be a continuous real valued function on $(X, d)$.
(a) Prove that there is an $x \in X$ for which
$$f(x) = \sup \{ f(t) : t \in X \}.$$ 
(b) Give an example of a set $K \subset \mathbb{R}$ and a function $f$ on $K$ for which the assertion (a) fails.

8. One form of completeness of the real numbers $\mathbb{R}$ is that every bounded increasing sequence converges. Use this property to prove that every Cauchy sequence in $\mathbb{R}$ converges.

9. Find the radius of convergence of each power series $\sum_{n=0}^{\infty} a_n x^n$.
   \[a) \ a_n = n \quad b) \ a_n = \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{n} & \text{if } n > 0 \end{cases} \quad c) \ a_n = \begin{cases} 1 & \text{if } n = 2^k, \text{ some } k \geq 0 \\ 0 & \text{otherwise} \end{cases}\]

10. Let $f$ be a function with domain $D \subset \mathbb{R}^2$, range $\mathbb{R}^2$, and defined by
$$f(x, y) = \left( \frac{x}{y}, \frac{y}{x} \right).$$
(a) What is the natural domain $D$ of $f$?
(b) The local inverse mapping theorem applies to $f$. Find the set $J$ for which the theorem guarantees a local inverse.