1. a) Determine whether the following series converge or diverge. Justify your answer.
\[ \sum_{k=1}^{\infty} \frac{1}{e^k}, \sum_{k=1}^{\infty} \frac{1}{3k+1}. \]

b) Prove the series
\[ \sum_{k=0}^{\infty} \frac{x^k}{k!} \]
converges uniformly on \([-a, a]\) for any positive real constant \(a\).

2. Let \(f_n(x) = e^{-nx}\) for \(n \geq 1\) and \(x \geq 0\).
   a) Show that \(f_n\) has a pointwise limit on \([0, \infty)\).
   b) Does \(f_n\) converge uniformly? Support your claim with a rigorous argument.

3. a) State the Monotone Convergence Theorem for sequences of reals numbers \(\langle a_n \rangle_{n=1}^{\infty}\).
   b) Let \(a_n = \sum_{k=1}^{n} \frac{1}{k} \frac{1}{3^k}\). Prove that \(\lim_{n \to \infty} (a_n)\) exists.

4. Suppose that \((x_n)\) is a Cauchy sequence in a compact metric space \(K\). Show directly using the definitions of "Cauchy sequence" and "compact set" that the sequence converges in \(K\).

5. a) Let \(X\) be a metric space with metric \(d\) and let \(f : X \to X\). Define what it means for \(f\) to be a contraction on \(X\).
   
   b) Use the Mean Value Theorem to show that if \(f : \mathbb{R} \to \mathbb{R}\) has a derivative satisfying \(|f'(x)| \leq \lambda\) for all \(x\) with \(0 \leq \lambda < 1\), then \(f\) is a contraction on \(\mathbb{R}\).

6. Suppose that \((f_n)\) is a sequence of continuous functions on \([0, 1]\) which converge to \(f\) on \([0, 1]\).
   a) If the convergence is uniform prove that
\[ \lim_{n \to \infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx. \]

   b) Give example in which convergence is not uniform and
\[ \lim_{n \to \infty} \int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx. \]

7. a) Let \(f\) be a continuously differentiable function from \(\mathbb{R}^2\) to \(\mathbb{R}\). Show that if there is a local minimum at \((0, 0)\) then we must have \(\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0\).
   
   b) If \(\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0\), does this imply that there is a local extrema at \((0, 0)\)? Prove it or give a counterexample.