1. An $n \times n$ real matrix $B$ is defined by: $B_{ij} = 1, \forall i, j = 1 \ldots n$.

(a) Find a basis for the column space of $B$.

(b) Find a basis for the null space of $B$.

(c) Show that the eigenvalues of $B$ are either 0 or $n$. Find an eigenvector corresponding to the eigenvalue $n$.

2. Let $V$ and $W$ be finite dimensional vector spaces and $T : V \to W$ be a linear transformation. $T$ is one-to-one iff $T(x) = T(y) \Rightarrow x = y, \forall x, y \in V$. $T$ is onto iff the range $R(T) = W$.

(a) Give an example (including a clear explanation) of a linear transformation which is:

(i) one-to-one but NOT onto

(ii) onto but NOT one-to-one.

(b) Suppose $\dim(V) = \dim(W)$. Then prove that $T$ is one-to-one iff $T$ is onto.

3. Consider the vector space $M_{2 \times 2}(C)$ of all $2 \times 2$ complex matrices. Let $A^*$ denote the adjoint of $A \in M_{2 \times 2}(C)$ and $Tr(A)$ denote its trace. Define $W \subset M_{2 \times 2}(C)$ to be the set all $2 \times 2$ matrices satisfying

$$W = \{X \in M_{2 \times 2}(C) : X^* = X, Tr(X) = 0\}.$$ 

(a) Prove that $W$ is a subspace of $M_{2 \times 2}(C)$.

(b) Show that a basis for $W$ is given by $\beta = \{\sigma_1, \sigma_2, \sigma_3\}$ where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

(c) Let $[X]_{\beta}$ denote the coordinates of a vector $X \in W$ with respect to the ordered basis $\beta$ given above. Prove that $\phi_{\beta} : W \to \mathbb{R}^3$ defined by $\phi_{\beta}(X) := [X]_{\beta}$ is an invertible linear transformation.

4. Let $X, Y \in W$ as defined in Problem 3.

(a) Prove that $<X, Y> = Tr(XY)/2$ defines an inner product on $W$.

(b) Let $\beta$ be the ordered basis for $W$ given in Problem 3. Show that

(i) $\beta$ is an orthonormal basis with respect to the above inner product.

(ii) $<X, Y> = <[X]_{\beta}, [Y]_{\beta}>'$ where $<, >'$ is the standard inner product on $\mathbb{R}^3$.

(c) Compute $<X, X>$ and deduce that every nonzero $X \in W$ is an invertible matrix.

over
5. (a) Let \( m(\lambda) \) and \( p(\lambda) \) denote, respectively, the minimal and characteristic polynomials associated with an \( n \times n \) matrix \( A \). Prove that the set of distinct zeros of \( m(\lambda) \) and \( p(\lambda) \) are the same.

(b) Suppose a \( 5 \times 5 \) matrix \( A \) has a minimal polynomial \( m(\lambda) = \lambda^2 + \lambda - 2 \).

(i) List the eigenvalues of \( A \) together with all possible algebraic multiplicities.

(ii) Show that \( A \) is invertible and express \( A^{-1} \) as a polynomial in \( A \).

6. (a) Suppose the characteristic polynomial of a matrix \( A \) is given by \( p(\lambda) = \lambda^2(\lambda - 1)^2 \). List all possible inequivalent (not similar) Jordan canonical forms \( J \) such that \( A = PJP^{-1} \).

(b) Given the matrix

\[
A = \begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
\end{pmatrix}
\]

Determine the Jordan canonical form \( J \) and the matrix \( P \) such that \( A = PJP^{-1} \).