1. Let $V$ and $W$ be vector spaces and $T : V \rightarrow W$ be a linear transformation.
   (a) If $V$ is finite dimensional, then show that the null space $N(T)$ and the range $R(T)$ are also finite dimensional, and $\dim(N(T)) + \dim(R(T)) = \dim(V)$.
   (b) If there exists a linear transformation $T : V \rightarrow W$ such that $N(T)$ and $R(T)$ are both finite dimensional, then prove that $V$ is finite dimensional.

2. Suppose $A$ is a $4 \times 4$ matrix with characteristic polynomial $p(x) = (x - 1)(x + 2)^2(x - 3)$.
   (a) Show that $A$ is invertible. Find the characteristic polynomial of $A^{-1}$.
   (b) Find the determinant and trace of $A$ and $A^{-1}$.
   (c) Express $A^{-1}$ as a polynomial in $A$. Explain your answer.

3. Let $P_n$ denote the vector space of real polynomials of degree at most $n$. Define the map $T : P_n \rightarrow P_n$ by
   $$T(p)(x) = \frac{d(xp)}{dx}, \quad p \in P_n.$$ 
   (a) Show that $T$ is linear and prove that $T$ is invertible.
   (b) Find an ordered basis $\beta$ for $P_n$ consisting of eigenvectors of $T$ and compute the matrix representation $[T]_\beta$.

4. Suppose $V$ is an inner product space, and let $\mathbf{w} \in V$ be a given unit vector. Define a linear transformation $T : V \rightarrow V$ by $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle \mathbf{w}$, $\forall \mathbf{v} \in V$.
   (a) Find explicitly the adjoint $T^*$ of $T$ and show that $T^* = T$.
   (b) Prove that $T$ is an orthogonal projection by showing that (i) $T^2 = T$ and (ii) $R(T)$ and $N(T)$ are orthogonal complements of each other.

5. Let $A$ be a real, $n \times n$, orthogonal matrix, i.e., $A^TA = AA^T = I$.
   (a) If $\lambda$ is a (complex) eigenvalue of $A$, then show that (i) $|\lambda| = 1$ and (ii) $\overline{\lambda}$ is also an eigenvalue.
   (b) If $n$ is odd, then prove that $A$ has at least one real eigenvalue.
   (c) When $n$ is even, give an example of the matrix $A$ which has no real eigenvalues.
6. (a) Suppose the characteristic polynomial of a matrix $A$ is given by $p(x) = x^2(x - 1)^2$. List all possible inequivalent (not similar) Jordan canonical forms $J$ such that $A = PJP^{-1}$.

(b) Given the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Determine the Jordan canonical form $J$ and the matrix $P$ such that $A = PJP^{-1}$.