1. Let \((S, d)\) be a metric space.
(a) Give the definition of a Cauchy sequence \(\{x_n\}\) in \((S, d)\).
(b) Prove that every convergent sequence in \((S, d)\) is a Cauchy sequence.
(c) Give an example of a metric space which has a Cauchy sequence that does not converge.

2. Let \(f_n(x) = x^n(1-x^n), \ x \in [0, 1]\).
(a) Show that \(f_n \to 0\) point-wise, for all \(x \in [0, 1]\). Justify your answer.
(b) Find the maximum of \(f_n(x)\) on \([0, 1]\). (Hint: take \(u = x^n\).
(c) Using your result of part (b) or otherwise, show that the sequence \(\{f_n\}\) does not converge uniformly on \([0, 1]\).

3. (a) Determine if each of the following series converge. Justify your answer.

\[
\begin{align*}
(i) & \ \sum_{n=0}^{\infty} \sqrt{n+1} - \sqrt{n} \\
(ii) & \ \sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^2} \\
(iii) & \ \sum_{n=1}^{\infty} \frac{n^n}{n!}
\end{align*}
\]

(b) Find the interval of convergence and the sum of the power series \(1 + x + x^2 + \cdots\).
(c) Use the result of part (a) to compute the sum \(\sum_{n=1}^{\infty} nx^n\). State any theorems you need to use to justify your computation.

4. (a) Let \(\{a_n(x)\}\) be a sequence of functions defined on an interval \(I \subseteq \mathbb{R}\). Then state and prove the Weierstrass M-test for uniform convergence of the series \(\sum_{n=0}^{\infty} a_n(x)\) on \(I\).
(b) Using the Weierstrass M-test or otherwise, prove that the series \(\sum_{n=0}^{\infty} \frac{\ln(nx)}{n^2}\) represents a continuous function on the interval \([1, 4]\).

5. (a) Give an example of a sequence of functions \(f_n : [0, 1] \to \mathbb{R}\) such that \(f_n \to 0\) point-wise on \([0, 1]\) but \(\int_0^1 f_n \to 0\).
(b) Let \(f_n : [a, b] \to \mathbb{R}\) be a sequence of continuous functions such that \(f_n \to f\) on \(C([a, b], \mathbb{R})\). Then prove that \(\int_a^b f_n \to \int_a^b f\).