Comprehensive Exam – Analysis (June 2013)

There are 5 problems, each worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) Suppose that $a_n \geq 0, b_n \geq 0$ are two non-negative sequences and that there exists $L > 0$ such that $\lim_{n \to \infty} \frac{a_n}{b_n} = L$. Then show that the series $\sum_n a_n$ and $\sum_n b_n$ either both converge or both diverge.

(b) Determine whether the following series converge. Justify your answers.
   
   (i) $\sum_{k=1}^{\infty} \frac{k + 3}{7k^2 + 8}$.  
   
   (ii) $\sum_{k=1}^{\infty} \frac{1}{k} \sin \left( \frac{1}{k} \right)$.

2. (a) Suppose the series $\sum_n a_n$ converges absolutely. Then show that $\sum_{n=1}^{\infty} a_n \cos(nx)$ converges uniformly for all $x \in \mathbb{R}$.

(b) Suppose that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on the interval $[a - \delta, a + \delta]$ for some $\delta > 0$, and $\lim_{x \to a} f_n(x) = c_n$. Prove that
   
   (i) $\sum_{n=1}^{\infty} c_n$ converges  
   
   (ii) $\lim_{x \to a} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} c_n$.

3. (a) Let $f(x)$ be a function defined on $[0, \infty)$ such that $f(0) = 0$ and the derivative $f'(x)$ is strictly increasing on $(0, \infty)$. Show that $g(x) = x^{-1} f(x)$ is strictly increasing on $(0, \infty)$.

(b) Let $f(x)$ be a continuous function on $[0, \infty)$ such that $\lim_{x \to \infty} f(x) = 0$. Then show that
   
   $\lim_{x \to \infty} e^{-x} \int_{0}^{x} f(t)e^{t} dt = 0$.

4. (a) For metric spaces $X, Y$ with metrics $d_X, d_Y$, let $X \times Y$ denote the set of ordered pairs $(x, y)$, with $x \in X$ and $y \in Y$. Show that $X \times Y$ is a metric space with metric
   
   $d((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2))$.

(b) Suppose $S_1$ and $S_2$ are metric spaces. If $f : S_1 \to S_2$ is a continuous function, and $K \subset S_1$ is compact, then prove that the image $f(K)$ is a compact subset of $S_2$. 
5. (a) A metric space $X$ with metric $d$ is called **sequentially compact** if every sequence $\{x_n\}$ from $X$ has a convergent subsequence. If $K$ is a closed subset of a sequentially compact metric space $X$, then prove that $K$ is sequentially compact.

(b) Suppose that $f: \mathbb{R}^N \to \mathbb{R}$ has partial derivatives at each $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$. Define

$$\nabla f := \begin{pmatrix}
\frac{\partial f}{\partial x_1} \\
\frac{\partial f}{\partial x_2} \\
\vdots \\
\frac{\partial f}{\partial x_N}
\end{pmatrix}.$$ 

Show that if $f$ has a local minimum at $x_0 \in \mathbb{R}^N$, then $\nabla f(x_0) = 0$. 