Master of Science Exam in Applied Mathematics
Analysis – August 19, 2005

There are 10 problems here. The best 7 will be used for the grade.

1. Consider the space of functions

\[ C = C([0,1], \mathbb{R}) = \{ f : f \text{ maps } [0,1] \to \mathbb{R} \text{ and } f \text{ is continuous} \} \]

and define

\[ d(f,g) := \sup\{|f(x) - g(x)| : x \in [0,1]\}. \]

(a) Show that \( d(f,g) \) is a metric on \( C \).
(b) Let

\[ \mathcal{F} := \{ f \in C : 0 \leq f(x) \leq 1 \text{ for } x \in [0,1] \}. \]

Show that \( \mathcal{F} \) is (i) bounded and (ii) closed as a set in the metric space \( C \) under the metric \( d(f,g) \).
(c) Define a sequence of functions \( \{ f_n \} \) in \( C \) by

\[ f_n(x) = x^n, \quad x \in [0,1]. \]

Show that there is no subsequence \( \{ f_{n_k} \} \) of the given sequence that converges in \( (C,d) \).

2. Define a sequence \( \{ a_n \} \) in \([0,1] \subset \mathbb{R}\) by \( a_n = \sin(n) \). Even though there seems to be no apparent pattern in the values of this sequence, it must have a convergent subsequence. State the relevant theory that proves the existence of such a convergent subsequence.

3. Define \( g_n : [0,1] \to \mathbb{R} \) by \( g_n(x) = e^{-nx^2} \).
   (a) Show that \( \lim_{n \to \infty} g_n(x) = 0 \) uniformly on \([0,1] \).
   (b) Prove in addition that \( \sum_{n=1}^\infty g_n(x) \) converges uniformly on \([0,1] \).

4. Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( g(x,y) = \sin(x/3) + \cos(y/3) \).
   (a) Show that the gradient vector of partial derivatives \( \nabla g = (\partial g/\partial x, \partial g/\partial y) \) satisfies \( \|\nabla g\| \leq 1/2 \) for all \((x,y) \in \mathbb{R}^2\), (the vector norm is the Euclidean norm).
   (b) The Mean Value Theorem asserts that if a function \( f : \mathbb{R}^2 \to \mathbb{R} \) has continuous partial derivatives on all of \( \mathbb{R}^2 \) then for all \((x_0,y_0), (x,y) \in \mathbb{R}^2 \) there exists \( \theta = \theta(x_0,y_0,x,y) \in (0,1) \) such that

\[ f(x,y) = f(x_0,y_0) + (\nabla f) \cdot (x-x_0,y-y_0), \]

where the dot product is indicated in the formula and where each partial derivative in the gradient vector \( \nabla f = (\partial f/\partial x, \partial f/\partial y) \) is evaluated at the point \((x_0 + \theta(x-x_0), y_0 + \theta(y-y_0)) \in \mathbb{R}^2 \). Conclude that

\[ |g(x,y) - g(x_0,y_0)| \leq (1/2)\|x-x_0,y-y_0\| \]

for all \((x_0,y_0), (x,y) \in \mathbb{R}^2 \).

(c) Define also \( h : \mathbb{R}^2 \to \mathbb{R} \) by \( h(x,y) = \sin(x/5) + \cos(y/5) \), and define the mapping \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( F(x,y) = (g(x,y), h(x,y)) \). Let \((a_0,b_0) = (0,0) \in \mathbb{R}^2 \) and inductively define \((a_{n+1},b_{n+1}) = F(a_n,b_n) \in \mathbb{R}^2 \). Verify that the mapping \( F \) on \( \mathbb{R}^2 \) is indeed a contraction and so conclude by the Contraction Mapping Theorem (check the hypotheses please) that the sequence \( \{(a_n,b_n)\} \) has a limit in \( \mathbb{R}^2 \).
5. Let \( f : [0, 1] \to \mathbb{R} \)
   (a) Define uniform continuity for \( f \) on \((0, 1]\).
   (b) Assume \( f \) is uniformly continuous. Let \( \{x_n\} \) be a Cauchy sequence in \((0, 1]\). Show that \( \{f(x_n)\} \) is a Cauchy sequence in \( \mathbb{R} \).

6. Define \( f \) on \( \mathbb{R} \) by
   \[
   f(x) = \begin{cases} 
   \frac{\sin x}{x} & \text{if } x \neq 0 \\
   1 & \text{if } x = 0 
   \end{cases}
   
   (a) Show that \( f \) is continuous on \( \mathbb{R} \).

   (b) Show that \( f'(0) \) exists and find \( f'(0) \).
   Hint: One approach is l'Hopital's rule.

7. Let \( K \) be a compact set in a metric space \((X, d)\) and let \( f \) be a continuous real valued function on \((X, d)\).
   (a) Prove that there is an \( x \in X \) for which
   \[
   f(x) = \sup \{f(t) : t \in X\}.
   
   (b) Give an example of a set \( K \subset \mathbb{R} \) and a function \( f \) on \( K \) for which the assertion (a) fails.

8. One form of completeness of the real numbers \( \mathbb{R} \) is that every bounded increasing sequence converges. Use this property to prove that every Cauchy sequence in \( \mathbb{R} \) converges.

9. Find the radius of convergence of each power series \( \sum_{n=0}^{\infty} a_n x^n \).
   \[
   \begin{array}{ll}
   a) & a_n = n \\
   b) & a_n = \begin{cases} 
   0 & \text{if } n = 0 \\
   \frac{1}{n} & \text{if } n > 0 
   \end{cases} \\
   c) & a_n = \begin{cases} 
   1 & \text{if } n = 2^k, \text{ some } k \geq 0 \\
   0 & \text{if otherwise}
   \end{cases}
   \end{array}
   
10. Let \( f \) be a function with domain \( D \subset \mathbb{R}^2 \), range \( \mathbb{R}^2 \), and defined by
    \[
    f(x, y) = \left( \frac{x}{y-x} \right).
    
    a) What is the natural domain \( D \) of \( f \)?
    b) The local inverse mapping theorem applies to \( f \). Find the set \( J \) for which the theorem guarantees a local inverse.