On the axiom of union

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Abstract In this paper, we study the union axiom of ZFC. After a brief introduction, we sketch a proof of the folklore result that union is independent of the other axioms of ZFC. In the third section, we prove some results in the theory $T:=\text{ZFC minus union}$. Finally, we show that the consistency of $T$ plus the existence of an inaccessible cardinal proves the consistency of ZFC.

Keywords Axiom of union · Transitive closure · Inaccessible cardinal

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1 Introduction

The purpose of this paper is to study the union axiom of ZFC. Recall that the axiom of union is the assertion that if $X$ is any set, then there exists a set $Y$ such that for all sets $x, x \in Y$ iff $x \in Z$ for some $Z \in X$. After briefly sketching the standard proof of the independence of this axiom, we proceed to show that quite a bit of set theory survives without appealing to union. Let $T$ be the theory ZFC minus union. After developing the rudimentary theory of the ordinals in $T$, we demonstrate (working in $T$) that every set is equinumerous with an ordinal. This allows us to define the cardinality of a set in the usual way. We use this fact to prove that various unions exist without appealing to the axiom of union. For example, if $S$ is any set for which the cardinalities of the elements of $S$ are bounded, then $\bigcup S$ exists (and hence arbitrary finite unions and arbitrary unions of countable sets can be proved to exist without appealing to the union axiom). In particular, if $\{|a| : a \in S\}$ contains but finitely many infinite cardinals, then
T proves that $\bigcup S$ exists. We also show that this is best-possible in the sense that $T$ does not prove the existence of a set $X$ such that $\{|a| : a \in X\}$ contains infinitely many infinite cardinals and $\bigcup X$ exists. After stating several equivalents of the union axiom in the theory $T$, we close the paper by showing that ZFC minus union is equiconsistent with ZFC modulo a large cardinal.

2 Independence of the union axiom

Gonzalez [4] uses the method of permutation models to show that union is independent of the other axioms of ZFC minus foundation and replacement. However, it turns out that union is independent of all other axioms of ZFC. This result is folklore. Since we could not locate a source for this result, we sketch the main ideas but omit the details.

Let $\kappa$ be an infinite cardinal and $X$ be a set. Recall that $X$ is hereditarily of cardinality less than $\kappa$ provided $|\text{TC}(X)| < \kappa$ (see Jech [5], Kunen [6], and Woodin [8]). Let us call $X$ pseudo-hereditarily of cardinality less than $\kappa$ provided:

(i) $|X| < \kappa$, and
(ii) If $y \in \text{TC}(X)$, then $|y| < \kappa$.

Let $H_\kappa$ denote the collection of all sets which are pseudo-hereditarily of cardinality less than $\kappa$. It can easily be shown that $H_\kappa$ is a set, and the proof of the following theorem is routine.

**Theorem 1** Let $\kappa$ be a singular strong limit (for example, $\beth_\omega$). Then $H_\kappa$ is a transitive model of all axioms of ZFC except for union.

We now state two corollaries of this result. In these corollaries, $U$ represents the union axiom.

**Corollary 1** If ZFC $- U$ is consistent, then ZFC $- U \nvdash U$.

**Proof** Suppose by way of contradiction that ZFC $- U$ is consistent and ZFC $- U \vdash U$. Then ZFC is consistent, yet ZFC $- U + \neg U$ is not. But Theorem 1 gives an interpretation of ZFC $- U + \neg U$ in ZFC, from which relative consistency follows (see, e.g., p. 63 of Shoenfield [7]).

**Corollary 2** If ZFC is consistent, then it cannot be shown in ZFC that ZFC is equiconsistent with ZFC $- U$.

**Proof** Suppose by way of contradiction that ZFC is consistent and that it can be shown (in ZFC) that the consistency of ZFC $- U$ implies the consistency of ZFC. By Theorem 1, ZFC proves the consistency of ZFC $- U$. Thus it can be shown that ZFC proves its own consistency, contradicting Gödel’s Second Incompleteness Theorem.

3 Some results in ZFC minus union

In this section, we work in the theory $T$ obtained by removing the union axiom from ZFC. One checks immediately that the union axiom plays no role in the statement of
On the axiom of union

the other axioms except (the inductive version of) the axiom of infinity. We recall that the successor of a set \( x \) is the set \( x \cup \{x\} \). In order for the axiom of infinity to have meaning when union is omitted, we need to know that the successor of a set can be proved to exist in \( T \). We prove this now.

**Lemma 1** \( T \vdash \text{if } b \text{ is any set, then } b \cup \{b\} \text{ exists.} \)

**Proof** Let \( b \) be any set. In \( \mathcal{P}(b) \), replace each singleton \( \{i\} \) by \( i \) and replace every other element, including \( \varnothing \), by \( b \). \( \Box \)

By an analogous argument, we can prove a slightly stronger result (which will be used shortly). We omit the easy proof.

**Lemma 2** \( T \vdash \text{if } x \text{ and } y \text{ are any sets, then } x \cup \{y\} \text{ exists.} \)

Recall that a set \( x \) is an ordinal provided \( x \) is transitive and the membership relation \( \in \) is a well-order on \( x \). We will need the following facts. We note that the standard proofs of these facts do not use the union axiom.

**Lemma 3** \( T \) proves the following:

1. If \( \alpha \) and \( \beta \) are ordinals and \( \alpha \subseteq \beta \), then \( \alpha \in \beta \).
2. No ordinal is an element of itself.
3. If \( \alpha \) and \( \beta \) are ordinals, then either \( \alpha \subseteq \beta \) or \( \beta \subseteq \alpha \).
4. If \( \alpha \) and \( \beta \) are distinct ordinals, then either \( \alpha \in \beta \) or \( \beta \in \alpha \).
5. Every transitive set of ordinals is an ordinal.
6. Every element of an ordinal is an ordinal.
7. If \( \alpha \) is an ordinal, then so is \( \alpha + 1 := \alpha \cup \{\alpha\} \) (note that \( \alpha + 1 \) exists by Lemma 1).

Now that we have these facts, we can prove that every set is equinumerous with some ordinal number. We note that the usual proof of this fact uses the transfinite recursion theorem (whose standard proof employs the union axiom). We give a modified proof which avoids the union axiom.

**Proposition 1** \( T \vdash \text{every set is equinumerous with an ordinal number.} \)

**Proof** Let \( X \) be an arbitrary set, and let \( G \) be a choice function for the collection of nonempty subsets of \( X \). Call a function \( f \) acceptable provided the domain of \( f \) is an ordinal, the range of \( f \) is contained in \( X \), and for every \( i \) in the domain of \( f \), one has:

\[
f(i) = G(X \setminus \{f(j) : j \in i\})
\]

We now establish the following claims:

**CLAIM 1:** If \( f \) and \( g \) are acceptable and \( i \in \text{dom}(f) \cap \text{dom}(g) \), then \( f(i) = g(i) \): Suppose by way of contradiction that there exist acceptable functions \( f \) and \( g \) and an \( i \in \text{dom}(f) \cap \text{dom}(g) \) with \( f(i) \neq g(i) \). Let \( i \) be the least such element in \( \text{dom}(f) \cap \text{dom}(g) \). It follows by the definition of acceptable and the leastness of \( i \) that \( f(i) = G(X \setminus \{f(j) : j \in i]\}) = G(X \setminus \{g(j) : j \in i]\}) = g(i) \) and this is a contradiction. Thus claim 1 is established.
CLAIM 2: If $f$ is an acceptable function and $i \neq j$ are in the domain of $f$, then $f(i) \neq f(j)$: Since $i \neq j$ and the domain of $f$ is an ordinal, we may assume that $i \in j$ by (4) and (6) of Lemma 3. We have $f(j) = G(X - \{f(x) : x \in j\})$. Since $i \in j$, it follows that $f(i) \neq f(j)$.

Call an element $x \in X$ special iff $x$ is in the range of some acceptable function.

CLAIM 3: If $x \in X$ is special, then there exists a unique ordinal $\alpha_x$ such that $f(\alpha_x) = x$ for some acceptable $f$: Indeed, suppose that $f(\alpha) = x$ and $g(\beta) = x$ for some ordinals $\alpha$ and $\beta$ and acceptable functions $f$ and $g$. Since the domains of $f$ and $g$ are ordinals, it follows from Claim 1 and (3) of Lemma 3 that we may assume that $f \subseteq g$. Thus $g(\alpha) = x = g(\beta)$. It now follows from Claim 2 that $\alpha = \beta$.

CLAIM 4: If $x \neq y$ are special, then $\alpha_x \neq \alpha_y$: Suppose that $\alpha_x = \alpha_y$. Then $f(\alpha_x) = x$ and $g(\alpha_x) = y$ for some acceptable functions $f$ and $g$. It now follows from Claim 1 that $x = y$.

We now let $S$ be the collection of special elements of $X$. By replacement, it follows that $F := \{(\alpha_x, x) : x \in S\}$ is a set. It follows from Claim 4 that $F$ is a function, and it follows again by replacement that the domain $D$ of $F$ is a set. We claim that $D$ is an ordinal. It suffices by (5) of Lemma 3 to prove that $D$ is transitive. Thus suppose that $\alpha \in \beta \in D$. Since $\beta \in D$, $f(\beta) \in X$ for some acceptable function $f$. Since $\alpha \in \beta$, it follows from the definition of acceptable that $f(\alpha) \in X$ as well. Hence $\alpha \in D$ and $D$ is an ordinal. Finally, we claim that $S = X$. Suppose by way of contradiction that $S \not\subseteq X$. Then note first that $F \cup \{(D, G(X - S))\}$ is a set by Lemma 2. It is also easy to check that $F \cup \{(D, G(X - S))\}$ is acceptable, and hence $G(X - S) \in S$, a contradiction. It follows from Claim 3 that $F$ is a bijection between $X$ and an ordinal, and the theorem is proved.

We use this result to show that $T$ proves the existence of Cartesian products. Recall the standard proof: If $x$ and $y$ are sets, then $x \times y \subseteq \mathcal{P}(x \cup y)$ (apply separation). Harvey Friedman has noted (unpublished) that one can prove the existence of $x \times y$ without using power set. For every $i \in y$, the set $x \times \{i\}$ exists by replacement. By replacement again, the set $S := \{x \times \{i\} : i \in y\}$ exists. By union, the set $\bigcup S = x \times y$ exists. We also remark that a somewhat different proof of the existence of $x \times y$ without power set appears in the literature in the context of Kripke-Platek set theory (see Proposition 1.3.2 of Barwise [1] for details). Apparently, it is not so easy to prove the existence of Cartesian products if one omits union.

**Proposition 2** $T \vdash$ if $x$ and $y$ are sets, then $x \times y$ is a set.

**Proof** Let $x$ and $y$ be arbitrary sets. By Proposition 1, we have $x \sim \alpha$ and $y \sim \beta$ for some ordinals $\alpha$ and $\beta$. We may assume without loss of generality that $\alpha \subseteq \beta$ by (3) of Lemma 3. Note that we may apply separation to obtain the existence of $\alpha \times \beta$ (since $\alpha \times \beta \subseteq \mathcal{P}(\mathcal{P}(\beta))$). It now follows from replacement that $x \times y$ exists. This completes the proof.

**Corollary 3** $T \vdash$ if $x$ and $y$ are sets, then the collection of all functions from $x$ to $y$ is a set.
First observe that if \( f : x \to y \) is a function, then \( f \subseteq x \times y \). Hence the collection of all functions from \( x \) to \( y \) is included in \( \mathcal{P}(x \times y) \). Applying separation yields the desired result. \( \square \)

Note that since every set is equinumerous with an ordinal, we may define cardinals in the theory \( T \) as usual: for any set \( x \), \(|x|\) is the least ordinal which is equinumerous with \( x \). We now state a sufficient condition (which is trivially also necessary) in order to prove in \( T \) that \( \bigcup x \) exists. We call a set \( S \) of ordinals bounded provided there exists an ordinal \( \alpha \) such that \( i \leq \alpha \) for all \( i \in S \).

**Theorem 2** \( T \vdash \text{if } x \text{ is a set, and if } \{|a| : a \in x\} \text{ is a bounded set of ordinals, then } \bigcup x \text{ exists.} \)

**Proof** We let \( x \) be a set and we assume that \( \{|a| : a \in x\} \) is a bounded set of ordinals. Thus there exists some ordinal \( \gamma \) such that \(|a| \leq \gamma \) for all \( a \in x \). Consider an arbitrary \( a \in x \). By assumption, \(|a| \leq \gamma \). Hence there exists a surjection from \( \{a\} \times \gamma \) onto \( a \) (clearly we may assume \( a \neq \emptyset \)). For each \( a \in x \), let \( S_a \) be the collection of all surjections from \( \{a\} \times \gamma \) onto \( a \). From Corollary 3 and separation, it follows that \( S_a \) is a set. By replacement, \( \{S_a : a \in x\} \) is a set. Finally, by the axiom of choice, we may pick a surjection \( G_a \) from \( \{a\} \times \gamma \) onto \( a \) for every \( a \in x \). We now define the formula \( P(c, d) \) by \( c = (a, i) \) for some \( a \in x \) and \( i \in \gamma \) and \( d = G_a(c) \). Since \( x \times \gamma \) is a set, it follows from replacement that \( \bigcup x \) is a set. This completes the proof. \( \square \)

The following corollary follows immediately.

**Corollary 4** \( T \vdash \text{if } x \text{ is a set and if } \{|a| : a \in x\} \text{ contains but finitely many infinite cardinals, then } \bigcup x \text{ exists.} \)

One interesting consequence of this corollary is that finite unions of sets can be proved to exist without appealing to the union axiom. We now show that this result is best possible.

**Proposition 3** Assume that \( ZF \) is consistent. Then \( T \not\models \exists x : \{|a| : a \in x\} \text{ contains infinitely many infinite cardinals and } \bigcup x \text{ exists.} \)

**Proof** Suppose that \( ZF \) is consistent. It is well-known (from Gödel’s work) that the consistency of \( ZF \) implies the consistency of \( ZFC \) plus the Generalized Continuum Hypothesis (GCH). Thus we may assume choice and GCH. In this setting, \( \exists \omega \models \mathcal{N}_\omega \).

Let \( M \models H_{\mathcal{N}_\omega} \). Then Theorem 1 implies that \( M \) is a model of \( T \). Let \( x \in M \). It is easy to see that \((x \text{ is a cardinal})^M \) iff \( x \) is a cardinal and \((x \text{ is an ordinal})^M \) iff \( x \) is an ordinal. Thus the infinite cardinals of \( M \) are precisely the cardinals \( \mathcal{N}_n \) where \( n \) is a natural number, and the ordinals of \( M \) are precisely the elements of \( \mathcal{N}_\omega \). Suppose by way of contradiction that \( T \vdash \text{there exists a set } x \text{ such that } \bigcup x \text{ is a set and } \{|y| : y \in x\} \text{ contains infinitely many infinite cardinals.} \]

Then in \( M \), there is a set \( x \) such that \( \bigcup x \) is a set and \( \{|y| : y \in x\} \) contains infinitely many infinite cardinals. But if this is the case, then \( \{|y| : y \in x\} \) must contain \( \mathcal{N}_n \) for arbitrarily large \( n \). In particular, \( \bigcup x \) must be \( \mathcal{N}_n \) for every natural number \( n \). However, the only infinite cardinals in \( M \) are the cardinals \( \mathcal{N}_n \) where \( n \) is a natural number, and so this is impossible. This completes the proof. \( \square \)
Our final result of this section lists several equivalents of union in the theory $T$.

**Theorem 3** $T \vdash$ the following are equivalent:

1. Every set of ordinals is bounded.
2. The union of an arbitrary set of ordinals exists.
3. The axiom of union.

**Proof** (1) $\Rightarrow$ (2): Suppose every set of ordinals is bounded. Let $S$ be an arbitrary set of ordinals, and let $\alpha$ be an upper bound for $S$. It now follows from separation that $\bigcup S$ exists.

(2) $\Rightarrow$ (3): Suppose that the union of an arbitrary set of ordinals exists, and let $x$ be an arbitrary set. By replacement, \{|a| : a \in x\} is a set. By our assumption, $\bigcup \{|a| : a \in x\} = \alpha$ exists. Thus by Theorem 2, it follows that $\bigcup x$ exists.

(3) $\Rightarrow$ (1): Assume the axiom of union, and let $S$ be any set of ordinals. It is well-known that $S$ is bounded by $\bigcup S$, and this completes the proof.

$\square$

4 A consistency result

In this section, we prove that if $T + “there exists an inaccessible cardinal”$ is consistent, then so is ZFC. We first recall the definition of an inaccessible cardinal. We remark that it follows from the results of the previous section that this definition is well-defined in $T$.

**Definition 1** A cardinal $\kappa$ is said to be inaccessible provided the following hold:

1. $\kappa > \aleph_0$.
2. $\kappa$ is a strong limit.
3. If $|I| < \kappa$ and \{$\alpha_i : i \in I$\} is a collection of cardinals all smaller than $\kappa$, then $\bigcup \{\alpha_i : i \in I\} < \kappa$ (since we need only separation, $T$ proves such unions exist).

We now prove our main theorem (compare to Corollary 2).

**Theorem 4** If $T + “there exists an inaccessible cardinal”$ is consistent, then so is ZFC.

**Proof** Assume that $T + “there exists an inaccessible cardinal”$ is consistent. Let $\kappa$ be an inaccessible cardinal. It is well-known (in ZFC) that the set $V_\kappa$ in the cumulative hierarchy is a model of ZFC (see for example Enderton [2, p. 255]). We prove that $V_\kappa$ exists without employing the union axiom. For the purposes of this proof, we call a function $f$ good iff $f$ satisfies the following:

1. The domain of $f$ is a subset of $\kappa + 1$.
2. Whenever $i \in \text{dom } f$, then $i \subseteq \text{dom } f$.
3. If $\varnothing \in \text{dom } f$, then $f(\varnothing) = \varnothing$.
4. If $\alpha + 1 \in \text{dom } f$, then $f(\alpha + 1) = \mathcal{P}(f(\alpha))$.
5. If $\alpha \in \text{dom } f$ is a nonzero limit ordinal, then $\bigcup \{f(i) : i \in \alpha\}$ exists and $f(\alpha) = \bigcup \{f(i) : i \in \alpha\}$.

$\square$
It is easily proved by induction that if \( f \) and \( g \) are good functions and \( i \) is an ordinal in the domain of both \( f \) and \( g \), then \( f(i) = g(i) \). Since the domain of every good function is a subset of \( \kappa+1 \), it follows by separation that the union of the domains of the good functions exists. By replacement (using the fact that any two good functions agree on the intersection of their domains), it follows that the union of all good functions exists. Call this union \( F \). By the above comments, \( F \) is a function. It is straightforward to check that \( F \) is good. We now claim that \( F \) is defined throughout \( \kappa+1 \). For suppose not, and let \( \alpha < \kappa+1 \) be the least ordinal where \( F \) is not defined. Clearly \( \alpha > 0 \). Suppose first that \( \alpha \) is a successor ordinal. Then \( \alpha = \beta + 1 \) for some ordinal \( \beta \). But then \( F \cup \{(\beta + 1, P F(\beta))\} \) is easily seen to be good (\( T \) proves that this union exists by Lemma 2). Thus \( \alpha \) is in the domain of \( F \), a contradiction. Suppose now that \( \alpha \) is a limit ordinal and consider \( F \cup \{(\alpha, \bigcup \{F(i) : i < \alpha\})\} \). In order for this expression to have meaning, we need to know that \( \bigcup \{F(i) : i < \alpha\} \) exists. Note that if \( i < \alpha \), then also \( i < \kappa \). It is straightforward to prove by induction (using the fact that \( \kappa \) is inaccessible) that \( |F(i)| < \kappa \) for every \( i < \alpha \). The set \( \{F(i) : i < \alpha\} \) exists by replacement. It follows now from Theorem 2 that \( \bigcup \{F(i) : i < \alpha\} \) exists. But then \( F \cup \{(\alpha, \bigcup \{F(i) : i < \alpha\})\} \) is good, and so \( \alpha \) is in the domain of \( F \), a contradiction. Thus \( F \) is defined throughout \( \kappa+1 \). It is now clear that \( F(\kappa) = V_\kappa \). The verification of the relativized axioms in \( V_\kappa \) proceeds just as in [2, p. 255]. This completes the proof.

In closing, we remark that \( T + \text{"There exists an inaccessible cardinal"} \) does not prove the axiom of union (by Theorem 1 with a singular strong limit cardinal greater than an inaccessible cardinal).

References