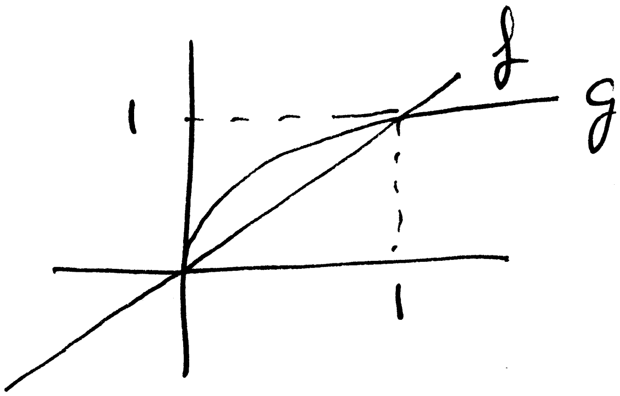
MATH 431, TEST 1, Fall 09

1.

(a) Let f and g be defined by $f(x) = x$ and $g(x) = \sqrt{x}$. Graph f and g on the same plot.



(b) Let a_n be a sequence of positive reals that converges to 0. Show that there is a natural N such that if $n \geq N$ then $a_n \leq \sqrt{a_n}$.

$\varepsilon = 1 \quad \exists N :$ if $n \geq N$ then $|a_n - 0| < 1$

$\Rightarrow |a_n| < 1 \Rightarrow 0 < a_n < 1 \Rightarrow a_n \leq \sqrt{a_n}$

2. Find a constants c such that for all x in $[-3, -2]$ we have

$$\frac{x}{1+x^2} \geq c.$$

$-3 \leq x \leq -2$

$4 \leq x^2 \leq 9$

$5 \leq 1 + x^2 \leq 10$

$\frac{1}{10} \leq \frac{1}{1+x^2} \leq \frac{1}{5}$

$-\frac{3}{5} \leq \frac{x}{5} \leq \frac{x}{1+x^2} \leq \frac{x}{10} \leq -\frac{2}{10}$

$c = -\frac{3}{5}$

3. Let a_n be a sequence of positive reals. Assume that $a_n^{1/n}$ converges to $1/2$.

(a) Show that there exists a natural N such that if $n \geq N$ then

$$a_n \leq (3/4)^n.$$

$\varepsilon = \frac{1}{4}$ $\exists N$: if $n \geq N$ $\left| a_n^{1/n} - \frac{1}{2} \right| < \frac{1}{4}$

$\frac{1}{4} < a_n^{1/n} < \frac{3}{4}$ $\forall n \geq N$

$\Rightarrow$ $a_n < \left(\frac{3}{4}\right)^n$ (since $x^n \uparrow$)

(b) Show that a_n converges.

$0 \leq a_n < \left(\frac{3}{4}\right)^n \leftarrow$ geom. with $|r|<1$

$\downarrow \quad \downarrow \quad \downarrow$ $a_n \to 0$ by

$0 \leq \quad \leq 0$ squeezing

4. Let a_n be a sequence of positive reals that converges to 0. Let

$$A = \{a_n; n \in \mathbf{N}\}.$$

(a) Show that A has a greatest lower bound.

$A \neq \emptyset$ since $a_1 \in A$

A is bounded below by 0 since $a_n \geq 0$ for all $n \geq 1$. The FP $\mathbb{R}$ applies A has a g.l.b.

greatest

4 (b) Find the lower bound of A

Take $b>0$, let $\varepsilon = b$ $\exists N$: if $n \geq N$

$|a_n - 0| < b \Rightarrow a_n < b \ \forall n \geq N$

$\Rightarrow$ b is not a lower bound. Hence a lower bound needs to be ≤ 0. Since 0 is a l.b. it is the g.l.b.

5. Assume that the sequence b_n converges to 1.

(a) Show that there exists N such that if $n \geq N$ then

$$|b_n| < \frac{5}{4}.$$

$\varepsilon = \frac{1}{4}$ $\exists N$: $\forall n \geq N$ we have $|b_n - 1| < \frac{1}{4}$

$|b_n| = |b_n - 1 + 1| \leq |b_n - 1| + |1|$

$< \frac{1}{4} + 1 = \frac{5}{4}$ for $n \geq N$

(b) Show that for any $r > 1$ there exists N such that if $n \geq N$ then

$$|b_n| < r.$$

$\varepsilon = r - 1 > 0$ $\exists N$: $n \geq N$

$|b_n - 1| < r - 1$

$|b_n| = |b_n - 1 + 1| \leq |b_n - 1| + 1$

$< r - 1 + 1 = r$

for $n \geq N$

6.

(a) Show that there is a constant K such that for all reals x

$$\frac{1}{1+x^2} \leq K.$$

$1+x^2 \geq 1$

$\frac{1}{1+x^2} \leq 1$ Take $K=1$

(b) Let a_n be a sequence of reals. Define the sequence b_n by

$$b_n = \frac{1}{1+a_n^2}.$$

Show that b_n has at least one subsequence that converges.

$|b_n| = b_n \leq 1 \Rightarrow b_n$ is bounded

$\overset{B.W.}{\Rightarrow}$ it has a subseq. that converges.

(c) State a generalization of (b) and prove it.

Assume f is bounded: $\exists M \cdot |f(x)| < M$ $\forall x \in \mathbb{R}$. Let x_n be a seq. then $f(x_n)$ has a convergent subseq.

Proof: $|f(x_n)| < M$ $\forall n \geq 1$ Hence $f(x_n)$ is bounded $\overset{B.W.}{\Rightarrow}$ $f(x_{n_k})$ converges for some x_{n_k}.