Quiz

Name: SOLUTIONS

Math 413/513 - Linear Algebra I - Fall 2007
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(1) Let $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 2 & 3 & -1 \end{pmatrix}$.

(a) Find the row-echelon form of the matrix and also the reduced row-echelon form.
(b) Show that the columns of A are linearly dependent [Hint: express the third column as a linear combination of the first two columns, using the reduced row echelon form of the matrix]
(c) Gaussian elimination can be used to determine whether a matrix M is invertible or not, by computing the reduced row echelon form of the augmented matrix $[M\ I]$. Explain why the reduced row echelon form of an invertible matrix must be I.
(d) Is the matrix A given above invertible?

(a) $\begin{pmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 2 & 3 & -1 \end{pmatrix} \xrightarrow[-2R_1+R_3 \Rightarrow R_3]{R_1+R_2 \Rightarrow R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 1 & -5 \end{pmatrix} \xrightarrow{-R_2 \Rightarrow R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 1 & -5 \end{pmatrix} \longrightarrow$

$\xrightarrow{-R_2+R_3 \Rightarrow R_3} \underbrace{\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{row echelon form}} \xrightarrow{-R_2+R_1 \Rightarrow R_1} \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow$ reduced row echelon form

(b) Elementary row operation preserve the linear combinations of the columns:

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 7 \\ -5 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow \boxed{c_1 = 7}$ and $\boxed{c_2 = -5}$

(c) Since M is invertible $\Rightarrow$ the linear system $Mx = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ has the unique solution $x = M^{-1}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1^{st}$ column of M^{-1}.
$\Rightarrow$ the augmented matrix $\left[M \,\middle|\, \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \right]$ has the reduced row echelon form $= [I \mid x]$. Ignoring the last column, it follows that the reduced row echelon form of M is I.

(d) Since the reduced form of A is not I (has third row $= 0$)
$\Rightarrow A$ is not invertible

(2) A square matrix $A = (a_{ij}) \in \mathcal{M}_3(\mathbb{F})$ is called upper triangular if $a_{ij} = 0$ for all $i > j$.
(a) Show that the product of two upper triangular matrices is upper triangular.
(b) Show that an upper triangular matrix is invertible if and only if its diagonal entries are all nonzero.
(c) If A is an upper triangular matrix which is also invertible, explain why its inverse must also be upper triangular.

(a) $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}$

$\Rightarrow AB = \begin{pmatrix} a_{11}b_{11} & DNC & DNC \\ 0 & a_{22}b_{22} & DNC \\ 0 & 0 & a_{33}b_{33} \end{pmatrix}$ = upper triangular

(DNC = do not care)

(b) Method 1 (using determinants)

Since $\det A = a_{11}a_{22}a_{33}$, A is invertible $\Leftrightarrow \det A \neq 0$
$\Leftrightarrow a_{11} \neq 0, a_{22} \neq 0, a_{33} \neq 0.$

Method 2 (without determinants)

We know that A is invertible iff the reduced row echelon form of A is equal to I. To obtain 1's on the diagonal, we need $a_{ii} \neq 0$. (necessary and sufficient condition!)

(c) Method 1 (with determinants)

A = invertible and upper triangular $\Rightarrow \det A = a_{11}a_{22}a_{33} \neq 0$

and $A^{-1} = \dfrac{1}{\det A}(A^T)^{*} = \dfrac{1}{\det A}\begin{pmatrix} a_{22}a_{33} & DNC & DNC \\ 0 & a_{11}a_{33} & DNC \\ 0 & 0 & a_{11}a_{22} \end{pmatrix}$ ✓

matrix of minors

Method 2 (without determinants)

If A = invertible $\Rightarrow$ $[A \mid I_3] \overset{RREF}{\sim} [I_3 \mid A^{-1}]$ (see problem (1c)),
The elementary row operations performed must be such that A^{-1} is also upper triangular.