SAMPLE EXAM II – Linear Algebra I - Fall 2007

You may use any theorems which were proved in class or in the textbook provided that you state them clearly. You may not use results which were proved as part of your homework unless you prove them again here.

- **Problem 1**

  (a) Show that the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \in \mathcal{M}_{3\times 3}(\mathbb{R})$ is NOT diagonazilable over the field of reals.

  (b) Show that the same matrix, viewed as $A \in \mathcal{M}_{3\times 3}(\mathbb{C})$ IS diagonalizable over the complex field.

  (c) Find an invertible matrix $Q \in \mathcal{M}_{3\times 3}(\mathbb{C})$ such that $Q^{-1}AQ$ is diagonal.

- **Problem 2**

  Show that if $T : V \to V$ is diagonalizable, then $V = N(T) \oplus R(T)$.

- **Problem 3**

  Let $T : V \to V$ be linear such that $T^k = 0$ for some positive integer k (such a T is called **nilpotent**).
  (a) Show that 0 is an eigenvalue of T and no other eigenvalues exists for T.
  (b) Using the Cayley-Hamilton theorem, show that $T^n = 0$, where $n = \dim(V)$.
  (c) Show that T is not diagonalizable (unless $T = 0$)

- **Problem 4**

  Let V be an inner product vector space over a scalar field F.
  (a) Show that $| < x, y > | = \|x\| \cdot \|y\|$ if and only if one of the vectors x or y is a scalar multiple of the other.
  (b) Similarly, show that $\|x + y\| = \|x\| + \|y\|$ if and only if $x = 0$ or $y = \lambda x$, for some $\lambda \in \mathbb{R}_+$.
  [See exercise 15, page 337]

- **Problem 5**

  Verify that the following vectors are orthogonal with respect to the standard inner product in $\mathbb{R}^4$, and then extend these to form an orthogonal basis in $\mathbb{R}^4$:

  $$v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

- **Problem 6**

  Let $V = C^\infty[-1, 1]$ be the space of complex valued functions, together wih the inner product $< f, g >= \int_{-1}^{1} f(t)\overline{g(t)}\, dt$.

  (a) Let W_1 be the subspaces of even functions ($f(-t) = f(t)$) and W_2 the subspace of odd functions ($f(-t) = -f(t)$). Show that $W_2 = W_1^\perp$ and $W_1 \oplus W_2 = V$. What is the orthogonal projection of a function $f \in V$ onto W_1?

  (b) Consider the linear transformations $Tf(t) = if'(t)$ and $Uf(t) = tf(t)$, restricted to the subspace W_0 of all functions satisfying $f(-1) = f(1)$. Show that both T and U are self-adjoint (that is $T^* = T$ and $U^* = U$).

- **Problem 7**

  Show that, for two linear transformations S and T from a finite dimensional inner product space V to itself, $R(S) \perp R(T)$ if and only if $S^*T = 0$.

- **Problem 8***

  Let $\mathcal{V} = \mathcal{M}_{n\times n}(\mathbb{C})$ and for $A, B \in \mathcal{V}$, define $< A, B >= \operatorname{tr}(AB^*)$.
  (a) Show that $< \cdot, \cdot >$ is an inner product on $\mathcal{V}$.
  (b) Show that if A is normal ($A^*A = AA^*$) , then

  $$\|A\|^2 = \operatorname{tr}(AA^*) = \sum_{i=1}^{n} |\lambda_i|^2$$

  where λ^i are the (not necessarily distinct) eigenvalues of A.
  (c) Find the orthogonal complement of the subspace of $\mathcal{V}$ consisting of all diagonal matrices.