Math 413/513 FALL 2007

SOLUTIONS
EXAM II – Linear Algebra

IN-CLASS PART

- **Problem 1**
  For each of the following statements, give either a short proof (if the statement is true) or a counterexample (if the statement is false)

  - *An $n \times n$ matrix having rank n is invertible.* TRUE

    Reason: Let A be an $n \times n$ matrix with maximum rank ($=n$). Then the left multiplication transformation $L_A : \mathbb{F}^n \to \mathbb{F}^n$ is onto and hence an isomorphism. This implies A is invertible.

  - *The matrix* $\begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 2 & 7 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ *is diagonalizable.* TRUE

    Reason: It is an upper triangular 4×4 matrix with distinct diagonal entries, hence it has 4 distinct eigenvalues. This implies diagonalizability (for each eigenvalue, the algebraic multiplicity equals one, and so does the geometric multiplicity).

  - *If T is a linear operator on a finite dimensional vector space V, then for any $v \in V$, the T-cyclic subspace generated by v is the same as the T-cyclic subspace generated by $T(v)$.* FALSE

    Reason: Let W be the T-cyclic subspace generated by a vector v. Then $W = \text{span}\{v, T(v), \ldots, T^{k-1}(v)\}$ for some $k(= \dim W)$. The T-cyclic subspace generated by $T(v)$ is $\text{span}\{T(v), T^2(v), \ldots T^{k-1}(v), \ldots\}$, which may not always equal W. For this it is enough to have $T^k(v) \in \text{span}\{T(v), T^2(v), \ldots T^{k-1}(v)\}$, for example if $T^k(v) = 0$. A concrete counterexample is $V = \mathbb{R}^2, T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

  - *For any nonempty subset S of an inner product space V, $S^\perp = [span(S)]^\perp$.* TRUE

    Reason: If $y \in S^\perp$, then y is orthogonal to all vectors in S. Hence y is orthogonal to all linear combinations of vectors in S, hence $y \in [\text{span}(S)]^\perp$. This proves the inclusion $S^\perp \subseteq [\text{span}(S)]^\perp$. The opposite inclusion is obvious (same argument can be involved).

- **Problem 2**
  Let A be an $m \times n$ matrix with rank n and B be an $n \times p$ matrix with rank p. Determine the rank of AB. Justify your answer.

  Solution: $\text{rank}(A) = n$ means that the left multiplication transformation $L_A : \mathbb{F}^n \to \mathbb{F}^m$ is one-to-one. (This can be seen from the dimension theorem: $\dim N(L_A) = \text{nullity}(L_A) = n - rank(L_A) = n - n = 0$.). Similarly, $\text{rank}(B) = p$ implies that $L_B : \mathbb{F}^p \to \mathbb{F}^n$ is one-to-one. Since the composition of two one-to-one maps is one-to-one, it follows that $L_{AB} = L_A L_B : \mathbb{F}^p \to \mathbb{F}^m$ is one-to-one, hence $\text{rank}(AB) = p$.

- **Problem 3**

  Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$.

  (a) State the Cayley-Hamilton theorem for the matrix A *and express* A^3 *as linear combination of* I_2 *and* A.

  Solution: We compute the characteristic polynomial for A to be $f(t) = t^2 - 4t - 5$. By the Cayley-Hamilton theorem, $A^2 - 4A - 5I_2 = O_2$, or $A^2 = 4A + 5I_2$. Multiplying by A both sides, we obtain

  $$A^3 = A(4A + 5I_2) = 4A^2 + 5A = 4(4A + 5I_2) + 5A = 21A + 20I_2.$$

  (b) Determine whether A *is diagonalizable or not. If yes, find an invertible matrix* Q *and a diagonal matrix* D *such that* $A = QDQ^{-1}$.

  Solution: The characteristic polynomial splits (factors into linear terms) as follows: $f(t) = (t+1)(t-5)$, therefore A has two distinct eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 5$. Hence A is diagonalizable. One finds an eigenvector $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ corresponding to $\lambda_1 = -1$ and an eigenvector $X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ corresponding to $\lambda_2 = 5$. We conclude that the matrices $D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ do the trick ($A = QDQ^{-1}$).

- **Problem 4**

  Given $V = \mathbb{R}^3$, *with the (standard) Euclidean inner product,*

  (a) Determine an orthonormal basis for the subspace $W = \{(x, y, z) \,|\, x - y + 3z = 0\}$

  Solution: First, we pick two linearly independent vectors in W (which will form a basis for W). Say $w_1 = (-3, 0, 1)$ and $w_2 = (0, 3, 1)$. Next, we orthogonalize this basis to obtain: $v_1 = w_1 = (-3, 0, 1)$ and $v_2 = w_2 - \frac{<w_2, v_1>}{\|v_1\|^2} v_1 = \frac{3}{10}(1, 10, 3)$. Finally, we orthonormalize this basis to obtain:

  $$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{10}}(-3, 0, 1), \quad u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{110}}(1, 10, 3).$$

  (b) Find the orthogonal projection of the vector $u = (2, 3, 1)$ *on the subspace* W *as in (a).*
  [Do not use the cross product!!!]

  Solution: Since $\{u_1, u_2\}$ is an orthonormal basis for W, the orthogonal projection of any vector $u \in \mathbb{R}^3$ is $< u, u_1 > u_1 + < u, u_2 > u_2$. For $u = (2, 3, 1)$ we compute the orthogonal projection of u onto W to be

  $$\frac{-5}{10}(-3, 0, 1) + \frac{35}{110}(1, 10, 3) = \frac{5}{11}(4, 7, 1).$$

- **Problem 5**

  Given a linear transformation T *on a finite dimensional vector space* V, *satisfying* $T^2 = T$,

  (1) *Show that the only possible eigenvalues of* T *are* $\lambda = 0$ *and* $\lambda = 1$, *and that* $N(T)$ *and* $R(T)$ *are the only possible eigenspaces.*

  Solution: If λ is an eigenvalue and $Tv = \lambda v$ for some $v \neq 0$, then $\lambda v = Tv = T^2 v = T(Tv) = T(\lambda v) = \lambda^2 v$, hence $\lambda^2 = \lambda$. This implies $\lambda = 0$ or $\lambda = 1$ are the only possible eigenvalues of T.
  If $\lambda = 0$ is an eigenvalue, the corresponding eigenspace is $E_0 = N(T - 0I) = N(T)$. If $\lambda = 1$ is an eigenvalue, then $E_1 = \{v \in V | Tv = v\} = R(T)$ (Last equality needs a bit of proof ...)

  (2) *Show that* T *is diagonalizable.*
  [Hint: Use that fact that the relation $T^2 = T$ *implies* $N(T) \oplus R(T) = V$ *and* T *is the projection on* $R(T)$ *along* $N(T)$*. You do not need to prove this here!]*

  Solution: From (a) , we know that $E_0 = N(T)$ and $E_1 = R(T)$ are the only possible eigenspaces of T. Since $E_0 \oplus E_1 = V$, it follows that we can construct β_0 a basis for E_0 and β_1 a basis for E_1, and then $\beta = \beta_0 \cap \beta_1$ is a basis for V, consisting of eigenvectors for T. (The cases when $\beta_0 = \emptyset$ or $\beta_1 = \emptyset$ are included above).

TAKE-HOME PART

- **Problem 1**
  Let $V = \mathbb{P}_2([-1,1],\mathbb{C})$ *be the vector space of polynomials of degree at most 2 with complex coefficients, with the 'standard' inner product* $< f, g > = \int_{-1}^{1} f(t)\overline{g(t)}\,dt$.

  Find the adjoint transformation T^* *of the linear transformation* $T : V \to V$*, defined by*
  $$T(f) = if' + 2f.$$

  Solution: We will use the fact that $[T^*]_\beta = [T]_\beta^*$ if β is an ORTHONORMAL basis for V. Such a basis can be obtained from the standard basis $\beta_0 = \{1, t, t^2\}$, by Gram Schmidt orthonormalization process (see Example 5 page 345):
  $$\beta = \{u_1, u_2, u_3\} = \{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}t, \sqrt{\frac{5}{8}}(3t^2 - 1)\}.$$
  Computing $T(\frac{1}{\sqrt{2}}) = 2\frac{1}{\sqrt{2}}$, $T(\sqrt{\frac{3}{2}}t) = i\sqrt{\frac{3}{2}} + 2\sqrt{\frac{3}{2}}t$ and $T(\sqrt{\frac{5}{8}}(3t^2-1)) = 3i\sqrt{\frac{5}{2}}t + 2\sqrt{\frac{5}{8}}(3t^2-1)$, we obtain $T(u_1) = 2u_1, T(u_2) = i\sqrt{3}u_1 + 2u_2, T(u_3) = i\sqrt{15}u_2 + 2u_3$. Hence,
  $$[T]_\beta = \begin{pmatrix} 2 & i\sqrt{3} & 0 \\ 0 & 2 & i\sqrt{15} \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow [T]_\beta^* = \begin{pmatrix} 2 & 0 & 0 \\ -i\sqrt{3} & 2 & 0 \\ 0 & -i\sqrt{15} & 2 \end{pmatrix}$$
  Finally, back in the standard basis β_0, $[T^*]_{\beta_0} = Q[T^*]_\beta Q^{-1}$, where $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\sqrt{\frac{5}{8}} \\ 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 3\sqrt{\frac{5}{8}} \end{pmatrix}$ is the change of basis matrix from β_0 to β.

- **Problem 2**
  (a) Let x, y *be linearly independent vectors in an inner product space* $V, < \cdot, \cdot >$ *such that* $\|x\| = \|y\| = 1$*. Prove that* $\|(1-t)x + ty\| < 1$*, for all* $t \in (0,1)$*.*

  Solution: We have
  $$\begin{aligned} \|(1-t)x + ty\|^2 &= < (1-t)x + ty, (1-t)x + ty > \\ &= \|(1-t)x\|^2 + \|ty\|^2 + 2Re < (1-t)x, ty > \\ &= (1-t)^2 + t^2 + 2t(1-t)Re < x.y > \\ &< (1-t)^2 + t^2 + 2t(1-t) = t^2 - 2t + 1 + t^2 + 2t - 2t^2 = 1 \end{aligned}$$
  Here we used the facts that $\|x\| = \|y\| = 1$ and the Cauchy Schwartz inequality $Re < x, y > \le | < x, y > | < \|x\|\|y\| = 1$.(The strict inequality is because of the assumption that x and y are not multiple of each other.)

  (b) Let $\|x\| = |x_1| + |x_2|$ *for* $x = (x_1, x_2) \in \mathbb{R}$*, be the taxi-cab norm on* $\mathbb{R}^2$ *(see general definition of a 'norm' on page 339). Use (a) to show that there is no inner product* $< \cdot, \cdot >$ *on* $\mathbb{R}^2$ *such that* $\|x\|^2 = < x, x >$*.*

  Solution: Let $x = (1,0)$ and $y = (0,1)$ and $t = \frac{1}{2}$. In the taxi cab norm, we can see that $\|x\| = \|y\| = 1$ but $\|\frac{1}{2}x + \frac{1}{2}y\| = 1$, so by part (a), this norm cannot originate from an inner product space.

- **Problem 3***

Let V be a finite dimensional vector space and T, U linear transformations which commute, i.e. $TU = UT$, and whose characteristic polynomials split.
(a) If λ is an eigenvalue for T, with corresponding eigenspace E_λ, show that E_λ is U-invariant.

Solution: Let $v \in E_\lambda$, so $Tv = \lambda v$. Then $T(Uv) = U(Tv) = U(\lambda v) = \lambda Uv$, which means $Uv \in E_\lambda$, so E_λ is U-invariant.

(b) Show that T and U must have at least one common eigenvector (in V).

Solution: Since the characteristic polynomial of T splits, it means that T has at least one eigenvalue λ. For this eigenvalue, the eigenspace E_λ is U-invariant, according to (a). Denote U', restriction of U to E_λ. Then the characteristic polynomial of U' divides the characteristic polinomial of U, hence splits. We conclude that U' admits at least one eigenvalue, and at least one eigenvector, which happens to be also in E_λ, hence it is a common eigenvalue for both T and U.

(c) If we assume, in addition, that T and U are both diagonalizable, show that T and U are simultaneous diagonalizable, that is there exists a common basis of eigenvectors for both T and U.
[Hint: If $E_1, \ldots, E_k$ denote the distinct eigenspaces of T, and $F_1, \ldots, F_l$ denote the distinct eigenspaces of U, such that $E_1 \oplus \ldots \oplus E_k = V = F_1 \oplus \ldots \oplus F_l$, show that $\sum_j W \cap F_j = W$ for each $W = E_i, i = 1, \ldots k$.]

Solution: Since T is diagonalizable, $V = E_1 \oplus \ldots E_k$. We will construct a basis for V consisting of eigenvectors for both T and U by constructing bases β_i for each E_i, $i = 1 \ldots k$, consisting of eigenvectors of U (and henceforth common eigenvectors for both T and U).

From now on we fix i and denote $W = E_i$, the i^{th} eigenspace of T. We claim that

$$W = W \cap F_1 \oplus \ldots \oplus W \cap F_l$$

Indeed, let $w \in W$. Since $w \in V = F_1 \oplus \ldots \oplus F_l$ (U is diagonlizable), it follows that $w = v_1 + \ldots v_l$ for some eigenvectors (for U) $v_j \in F_j$. By the result proved in HMW (Exercise 23, Sect 5.4), (you may include the proof here) since W is U-invariant, and $v_1 + \ldots + v_l = w \in W$, it follows that each $v_j \in W$, hence $v_j \in W \cap F_j$, $j = 1 \ldots l$. This proves that $w \in W \cap F_1 \oplus \ldots \oplus W \cap F_l$. Since the opposite inclusion is obvious, the claim is proved.

To construct a basis for $W (= E_i)$ consisting of eigenvectors for U (hence common for both T and U), it is enough to pick an arbitrary basis (say β_{ij}) for each $W \cap F_j$ and take their union after j:

$$\beta_i = \bigcup_{j=1}^{l} \beta_{ij} \text{ is a basis for } E_i, \text{ consisting of eigenvectors for both } T \text{ and } U$$

.
As was mentioned before, $\beta = \bigcup_{i=1}^{k} \beta_i$ is then the desired basis for V, consisting of eigenvectors for both T and U, which proves that T and U are simultaneous diagonalizable.