SOLUTIONS TO SAMPLE FINAL

MATH 136 – SPRING 2008

1. (a) $\int \arctan 4t \, dt$. First, use substitution $x = 4t$, $dx = 4\,dt$ to get $\int \arctan 4t \, dt = \frac{1}{4}\int \arctan x \, dx$. Next, use integration by parts

$$u = \arctan x \qquad dv = dx$$
$$du = \frac{1}{1+x^2}dx \qquad v = x$$

$$= \frac{1}{4}\int \underbrace{\arctan x}_{u} \underbrace{dx}_{dv} = \frac{1}{4}\left(x \arctan x - \int x \frac{1}{1+x^2} dx\right) = \frac{1}{4}\left(x \arctan x - \frac{1}{2}\ln(1+x^2)\right)$$

Hence

$$\int \arctan 4t \, dt = \frac{1}{4}\left[4t \arctan 4t - \frac{1}{2}\ln(1+(4t)^2)\right] = \boxed{t \arctan 4t - \frac{1}{8}\ln(1+16t^2)}$$

(b) $\int_0^1 y^2\sqrt{1+y^3}\,dy$. Use substitution $u = 1+y^3$, $du = 3y^2 dy$

$$= \frac{1}{3}\int_1^2 \sqrt{u}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big|_1^2 = \frac{2}{9}u^{3/2}\Big|_1^2 = \boxed{\frac{2}{9}\left(2^{3/2}-1\right)}$$

2. (a) $\int (1+\cos\theta)^2 d\theta = \int 1 + 2\cos\theta + \cos^2\theta \, d\theta = \theta + 2\sin\theta + \int \cos^2\theta \, d\theta$

$$= \theta + 2\sin\theta + \int \frac{1}{2}(1+\cos 2\theta)\,d\theta = \theta + 2\sin\theta + \frac{1}{2}\theta + \frac{\sin 2\theta}{4} + C$$

$$= \boxed{\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4} + C}$$

(b) $\int \frac{\sqrt{1+x^2}}{x}\,dx$. Use substitution $u=\sqrt{1+x^2}$

$\Rightarrow u^2 = 1+x^2 \Rightarrow x^2 = u^2-1 \Rightarrow 2x\,dx = 2u\,du$

$$= \int \frac{\sqrt{1+x^2}}{x^2}\,x\,dx = \int \frac{u}{u^2-1}\,u\,du = \int \frac{u^2}{u^2-1}\,du$$

$$= \int \frac{u^2-1+1}{u^2-1}\,du = \int 1 + \frac{1}{u^2-1}\,du = u + \int \frac{1}{(u-1)(u+1)}\,du$$

Partial fractions

$$= u + \frac{1}{2}\int \frac{1}{u-1} - \frac{1}{u+1}\,du = u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + c$$

$$= \sqrt{1+x^2} + \frac{1}{2}\ln\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + c.$$

Note: Alternative method, use substitution $x = \tan\theta$, $dx = \frac{1}{\cos^2\theta}\,d\theta$

to obtain $\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{1}{\sin\theta\cos^2\theta}\,d\theta = \int \frac{\sin\theta}{\sin^2\theta\cos^2\theta}\,d\theta$

and then use again a substitution $u = \cos\theta$, $du = -\sin\theta\,d\theta$

$$= -\int \frac{1}{u^2(1-u^2)}\,du = \int \frac{1}{u^2(u^2-1)}\,du = \text{(method of partial fractions)}$$

$$= \int \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1} + \frac{D}{u-1}\,du = \ldots$$

3 (a) $\displaystyle\int_1^\infty \frac{1}{(2x+1)^3}\,dx$ $\qquad 2x+1=u \Rightarrow du = 2\,dx$

$$= \int_3^\infty \frac{1}{u^3}\,\frac{1}{2}du = \frac{1}{2}\int_3^\infty \frac{1}{u^3}\,du \quad \text{(convergent since } p=3>1\text{)}$$

$$= \frac{1}{2}\,\frac{u^{-2}}{-2}\Big|_3^\infty = -\frac{1}{4}\left(\frac{1}{u^2}\right)\Big|_3^\infty = -\frac{1}{4}\left(0-\frac{1}{9}\right) = \boxed{\frac{1}{36}}$$

(b) $\displaystyle\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx$

Use int. by parts: $\quad u=\ln x \quad dv = \frac{1}{\sqrt{x}}dx$, $\quad du = \frac{1}{x}dx \quad v = 2\sqrt{x}$

$$\int \frac{\ln x}{\sqrt{x}}dx = \int \underbrace{\ln x}_{u}\,\underbrace{\frac{1}{\sqrt{x}}dx}_{dv} = 2\ln x\sqrt{x} - \int 2\sqrt{x}\,\frac{1}{x}dx$$

$$= 2\ln x\sqrt{x} - 2\int \frac{1}{\sqrt{x}}dx = 2\ln x\sqrt{x} - 4\sqrt{x} + C$$

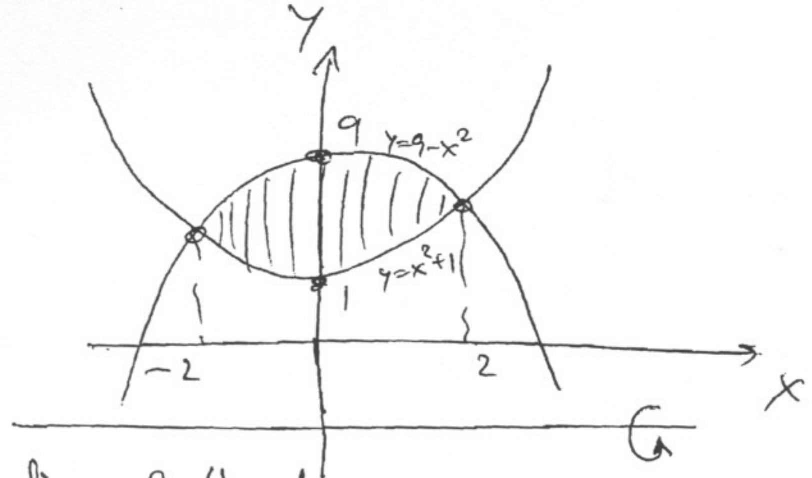
$$\Rightarrow \int_0^4 \frac{\ln x}{\sqrt{x}}dx = \left(2\ln x\sqrt{x} - 4\sqrt{x}\right)\Big|_0^4 =$$

$$= \left(2\ln 4\sqrt{4} - 4\sqrt{4}\right) - \lim_{x\to 0^+}\left(2\ln x\sqrt{x} - 4\sqrt{x}\right)$$

But $\displaystyle\lim_{x\to 0^+} \ln x\sqrt{x} = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} \underset{\text{l'Hôpital}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}\frac{1}{x\sqrt{x}}} = 0$

$\Rightarrow$ integral converges and $\displaystyle\int_0^4 \frac{\ln x}{\sqrt{x}}dx = 4\ln 4 - 4\sqrt{4} = \boxed{8\ln 2 - 8}$

④



$$r_{out} = 9 - x^2 + 1 = 10 - x^2$$
$$r_{in} = x^2 + 1 + 1 = x^2 + 2$$

Intersection of the two curves:

$$x^2 + 1 = 9 - x^2 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$$\text{Volume} = \int_{-2}^{2} \pi \left[r_{out}^2 - r_{in}^2\right] dx = \int_{-2}^{2} \pi \left((10 - x^2)^2 - (x^2+2)^2 \right) dx$$

$$= \pi \int_{-2}^{2} \left(100 - 20x^2 + x^4 - (x^4 + 4x^2 + 4)\right) dx$$

$$= \pi \int_{-2}^{2} 96 - 24x^2 \, dx = 2\pi \int_{0}^{2} 96 - 24x^2 \, dx =$$

$$= 2\pi \left(96x - 8x^3\right) \Big|_{x=0}^{2} = 2\pi \left(96 \cdot 2 - 8 \cdot 8\right) = \underline{256\pi}.$$

⑤ $$\text{Length} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \int_0^1 \sqrt{1 + (e^x)^2}\, dx = \int_0^1 \sqrt{1 + e^{2x}}\, dx$$

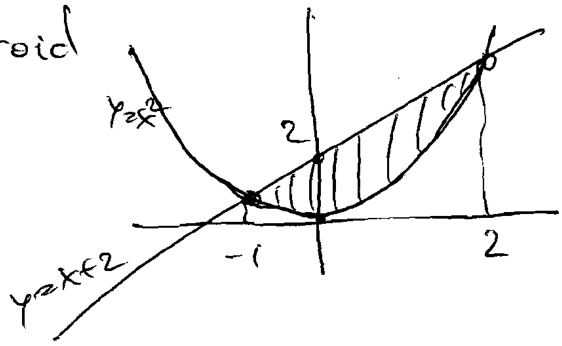
Use subs

$$u = \sqrt{1 + e^{2x}} \Rightarrow u^2 = 1 + e^{2x} \Rightarrow u^2 - 1 = e^{2x}$$

$$\Rightarrow 2x = \ln(u^2 - 1) \Rightarrow x = \frac{1}{2}\ln(u^2 - 1) \Rightarrow dx = \frac{u}{u^2 - 1}\, du$$

$$\Rightarrow \int_0^1 \sqrt{1 + e^{2x}}\, dx = \int_{\sqrt{2}}^{\sqrt{1+e^2}} u \cdot \frac{u}{u^2 - 1}\, du = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{u^2}{u^2 - 1}\, du \overset{\text{sd}(2b)}{=} u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| \Bigg|_{\sqrt{2}}^{\sqrt{1+e^2}}$$

6 Centroid



Intersection points $x^2 = x+2 \Rightarrow x^2-x-2=0$

$(x-2)(x+1)=0$

$x=2,\ x=-1$

$$\rightarrow A = \int_{-1}^{2} (x+2)-x^2\,dx = \ldots = 9/2$$

$$\bar{x} = \frac{1}{A}\int_{-1}^{2} x\left((x+2)-x^2\right)dx = \frac{1}{2}$$

$$\bar{y} = \frac{1}{2A}\int \left[(x+2)^2-(x^2)^2\right]dx = \frac{8}{5}$$

7 (a) $\{a_n\}$ is decreasing. One can see this by studying the function $f(x)=\frac{x+1}{2x^2-3}$. $f'(x) = \frac{1(2x^2-3)-(x+1)\cdot 4x}{(2x^2-3)^2} = \frac{-2x^2-4x-3}{(2x^2-3)^2} < 0$

$\Rightarrow f$ is decreasing on $[2,\infty)$.

(b) By alt series test, since $\lim_{n\to\infty}\frac{n+1}{2n^2-3}=0$ and (a)

$\Rightarrow \sum (-1)^n \frac{n+1}{2n^2-3}$ is convergent.

Since $\sum_{n\geq 2}\left|(-1)^n\frac{n+1}{2n^2-3}\right| = \sum_{n=2}^{\infty}\frac{n+1}{2n^2-3} \geq \sum_{n\geq 2}\frac{n}{2n^2} = \sum_{n=2}^{\infty}\frac{1}{2n} = +\infty$

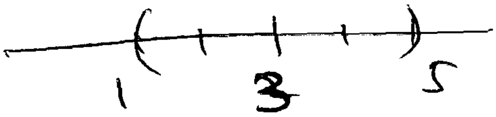
$\Rightarrow$ $\boxed{\sum (-1)^n \frac{n+1}{2n^2-3} \text{ is conditionally convergent!!}}$

(8) $$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2 2^n}$$

Radius of conv: Use Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{(n+1)^2 2^{n+1}}}{\frac{(x-3)^n}{n^2 2^n}} \right| = \left| \frac{x-3}{2} \right| < 1$$

$\Rightarrow |x-3| < 2 \Rightarrow \boxed{R=2}$



Interval contains $(1,5)$

$x=1 \Rightarrow \sum_{n=1}^{\infty} \frac{(1-3)^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ = abs conv.

$x=5 \Rightarrow \sum_{n=1}^{\infty} \frac{(5-3)^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ (Conv)

$\rightarrow$ Interval is $[1,5]$

(9) see class notes (Friday, May 9)

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(11) Length of a polar curve $r = \theta^2,\ 0 \le \theta \le 2\pi$

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$$

$$= \int_0^{2\pi} \theta\sqrt{\theta^2+4}\, d\theta = \frac{1}{2}\int_0 \sqrt{\theta^2+4}\, d(\theta^2+4) = \frac{1}{2}\cdot\frac{2}{3}(\theta^2+4)^{3/2}\Big|_0^{2\pi}.$$

12 (a) $r = 2\sin\theta + 2\cos\theta \quad / \cdot r$

$r^2 = 2r\sin\theta + 2r\cos\theta$

$x^2 + y^2 = 2y + 2x$

$\Rightarrow x^2 - 2x + y^2 - 2y = 0$

$\Rightarrow x^2 - 2x + 1 + y^2 - 2y + 1 = 2$

$\Rightarrow (x-1)^2 + (y-1)^2 = 2$

$\Rightarrow$ Circle centered at $(1,1)$ with radius $\sqrt{2}$.

(b) $r = \dfrac{1}{1+2\cos\theta} \Rightarrow r(1+2\cos\theta) = 1$

$\Rightarrow r + 2r\cos\theta = 1$

$\Rightarrow \sqrt{x^2+y^2} + 2x = 1$

$\Rightarrow \sqrt{x^2+y^2} = 1 - 2x$

$\Rightarrow x^2 + y^2 = (1-2x)^2$

$= 1 - 4x + 4x^2$

$\Rightarrow 3x^2 - 4x + 1 = y^2$

$\rightarrow$ hyperbola.