Friday, Mar 14, 2008 SOLUTIONS STUDENT NAME: ____________

EXAM III – MATH 136, SPRING 2008

READ EACH PROBLEM CAREFULLY!
You need to provide reasonable explanation for your answers in order to get credit for your work!!!!
The exam has 8 problems on 4 pages! Turn in all pages!

- **Problem 1**
  Determine whether the sequence below is convergent or divergent. If convergent, find its limit.

$$a_n = \frac{\cos^2 n}{2^n}, \quad n \geq 1$$

Since $0 \leq \cos^2 n \leq 1 \Rightarrow 0 \leq \frac{\cos^2 n}{2^n} \leq \frac{1}{2^n}$. Also $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

By squeezing theorem $\Rightarrow \lim_{n\to\infty} \frac{\cos^2 n}{2^n} = 0$.

- **Problem 2**
  Test the series for convergence or divergence. Explain!

$$\sum_{n=1}^{\infty} \frac{n^2}{3n-1}$$

Since $\lim_{n\to\infty} \frac{n^2}{3n-1} = +\infty$ (hence $\neq 0$), by the test for divergence we conclude that $\sum_{n=1}^{\infty} \frac{n^2}{3n-1}$ is divergent.

- **Problem 3**
  Determine whether the series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$$

This is an alternating series $\sum_{n\geq 1}^{\infty}(-1)^n b_n$, with $b_n = \frac{n}{n^2+1}$ and $\{b_n\}$ is a decreasing sequence convergent to 0.

$\left[f(x) = \frac{x}{x^2+1} \right.$ has $f'(x) = \frac{1\cdot(x^2+1) - x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1 \Rightarrow$

$\Rightarrow f$ is decreasing function on $[1,\infty) \Rightarrow \{b_n\}$ decreasing $\left. \right]$.

By alternating series test, $\sum (-1)^n \frac{n}{n^2+1}$ is convergent.

But $\sum_{n\geq 1}^{\infty} \left|(-1)^n \frac{n}{n^2+1}\right| = \sum_{n\geq 1} \frac{n}{n^2+1}$ is divergent by the limit comparison with $\sum_{n\geq 1}^{\infty} \frac{n}{n^2} = \sum_{n\geq 1}^{\infty} \frac{1}{n} = +\infty$

$\left[\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{n}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0 \right]$

Conclusion: Series $\sum_{n\geq 1}^{\infty} (-1)^n \frac{n}{n^2+1}$ is conditionally convergent!

- **Problem 4**
  Using the integral test, determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

is convergent or divergent.

We consider the improper integral

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx \underset{\left(\substack{\text{Subs. } u = \ln x \\ du = \frac{1}{x}dx}\right)}{=} \int_{\ln 2}^{\infty} \frac{1}{u}\,du =$$

$$= (\ln u)\Big|_{\ln 2}^{\infty} = +\infty \Rightarrow \int_2^{\infty} \frac{1}{x \ln x}\,dx \text{ is divergent}$$

In addition $f(x) = \frac{1}{x \ln x}$ is decreasing to 0 as $(x \to \infty)$.

Hence, by the integral test,

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ is divergent!}$$

- **Problem 5**

Determine if the geometric series below is convergent or divergent. If convergent, find its **sum**.

$$\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{2^{2n+1}}$$

We rewrite

$$\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{2^{2n+1}} = \sum_{n=0}^{\infty}(-1)^n\frac{3^n}{4^n\cdot 2} = \sum_{n=0}^{\infty}\frac{1}{2}\left(-\frac{3}{4}\right)^n$$

. geometric series with ratio $r=-\frac{3}{4}$.

$$= \frac{\frac{1}{2}}{1-\left(-\frac{3}{4}\right)} = \frac{\frac{1}{2}}{1+\frac{3}{4}} = \frac{\frac{1}{2}}{\frac{7}{4}} = \boxed{\frac{2}{7}}.$$

$\Rightarrow$ Convergent!

- **Problem 6**

Find the **radius of convergence** and **interval of convergence** of the power series

$$\sum_{n=1}^{\infty}\frac{(x-1)^n}{n^2 5^n}$$

Ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{(x-1)^{n+1}}{(n+1)^2 5^{n+1}}}{\frac{(x-1)^n}{n^2 5^n}}\right| = \lim_{n\to\infty}\frac{|x-1|}{5}\frac{n^2}{(n+1)^2} = \frac{|x-1|}{5}$$

By ratio test, series is (abs) conv if $\frac{|x-1|}{5}<1$

divergent if $\frac{|x-1|}{5}>1$

$\Rightarrow$ $|x-1|<5$ $\Rightarrow$ radius of convergence $\boxed{R=5}$.

Centered at $x=1$ $\Rightarrow$ Interval of convergence contains: $(-4,6)$

For $x=-4$ $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-5)^n}{n^2 5^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$

For $x=6$ $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{n^2 5^n} = \sum_{n=1}^{\infty}\frac{1}{n^2}$

abs. conv. $\left(\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty\right)$ p-test, $p=2>1$

$\Rightarrow$ Interval of convergence $\boxed{I=[-4,6]}$

- **Problem 7**

(a) Find the Maclaurin series of the function $f(x) = \ln(1+x^3)$ and determine its radius of convergence.

(b) Express the indefinite integral below as a power series.

on next page

$$\int \frac{\ln(1+x^3)}{x}\,dx$$

$$f'(x) = \frac{3x^2}{1+x^3}$$

Start with $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots \quad (|x|<1)$

$\Rightarrow \frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \dots + (-1)^n x^{3n} + \dots \quad (|x|<1)$

$\Rightarrow \frac{3x^2}{1+x^3} = 3x^2 - 3x^5 + 3x^8 - 3x^{11} + \dots + 3(-1)^n x^{3n+2} + \dots$

$\Rightarrow f(x) = \ln(1+x^3) = \int \frac{3x^2}{1+x^3}dx = x^3 \quad \frac{x^6}{2} + \frac{x^9}{3} - \dots + \frac{3(-1)^n}{3n+3} x^{3n+3} + \dots$

$$\boxed{\ln(1+x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{n+1}}$$

Radius of convergence $R = 1$

- **Problem 8**

Find the Taylor series for the function

$$f(x) = \frac{1}{1-2x}, \qquad \text{centered at } a = 1.$$

Determine its radius of convergence and interval of convergence.

$f(x) = \frac{1}{1-2x}$

$f'(x) = 2\frac{1}{(1-2x)^2}$

$f''(x) = 2^2 \frac{2}{(1-2x)^3}$

$\vdots$

$f^{(n)}(x) = 2^n \frac{n!}{(1-2x)^{n+1}} \Rightarrow \boxed{f^{(n)}(1) = 2^n \frac{n!}{(-1)^{n+1}}}$

$\vdots$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= \sum_{n=0}^{\infty} 2^n \frac{n!\,(-1)^{n+1}}{n!}(x-1)^n$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} 2^n (x-1)^n}$$

By ratio test

$$\lim_{n\to\infty} \left| \frac{2^{n+1}(x-1)^{n+1}}{2^n(x-1)^n} \right| =$$

$= 2|x-1| < 1 \Rightarrow$

Radius of convergence $\boxed{R = \frac{1}{2}}$

interval of convergence $= \left(\frac{1}{2}, \frac{3}{2}\right)$

(b) $\ln(1+x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{n+1}$

$\rightarrow \frac{\ln(1+x^3)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{n+1}$

$\Rightarrow \int \frac{\ln(1+x^3)}{x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{(n+1)(3n+3)} =$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{3(n+1)^2}.$