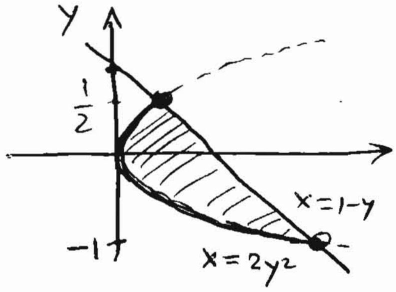
SOLUTIONS

Friday, Mar 14, 2008 ~~STUDENT NAME:~~

EXAM II – MATH 136, SPRING 2008

READ EACH PROBLEM CAREFULLY! To get full credit, you must show all work!
The exam has 8 problems on 4 pages! Turn in all pages!
NO GRAPHING CALCULATORS ALLOWED!



- **Problem 1**
Find the area of the region bounded by the curves

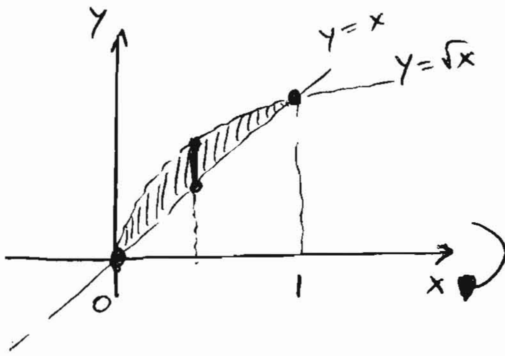
$$x = 2y^2, \quad x + y = 1.$$

Intersection points: $2y^2+y=1 \Rightarrow 2y^2+y-1=0$
$\Rightarrow (2y-1)(y+1)=0 \Rightarrow \boxed{y=-1}$ and $\boxed{y=\frac{1}{2}}$

$$\text{Area} = \int_{-1}^{1/2} [(1-y)-2y^2]\,dy = \int_{-1}^{1/2} (1-y-2y^2)\,dy = \left(y-\frac{y^2}{2}-\frac{2}{3}y^3\right)\Big|_{-1}^{1/2} = \frac{1}{2}-\frac{1}{8}-\frac{2}{3}\cdot\frac{1}{8}-$$
$$-\left(-1-\frac{1}{2}+\frac{2}{3}\right) = \frac{4}{3}-\frac{5}{24} = \frac{27}{24} = \boxed{\frac{9}{8}}$$

- **Problem 2**
Find the volume of the solid obtained by rotating the region bounded by the curves

$$y = x, \quad y = \sqrt{x}, \qquad \text{about the } x\text{-axis.}$$



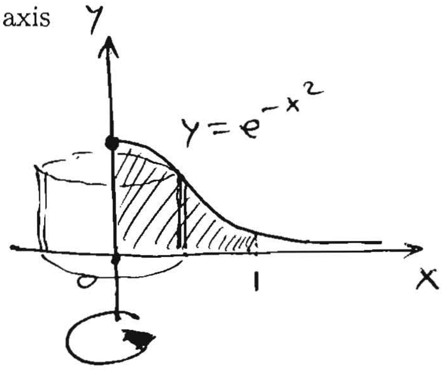
Intersection points
$y=x=\sqrt{x} \Rightarrow \boxed{x=0}$ or $\boxed{x=1}$

Using the method of washers,

$$\text{Vol} = \int_0^1 \pi\left[(\sqrt{x})^2 - x^2\right] dx$$
$$= \pi\int_0^1 x - x^2\, dx$$
$$= \pi\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^1$$
$$= \pi\left(\frac{1}{2}-\frac{1}{3}\right) = \boxed{\frac{\pi}{6}}$$

- **Problem 3**
Using the method of cylindrical shells, find the volume of the solid obtained by rotating the region bounded by the curves

$$y = e^{-x^2}, \quad y = 0, \quad x = 0, \quad x = 1, \qquad \text{about the } y\text{-axis}$$



$$\text{Volume} = \int_0^1 2\pi x \, e^{-x^2} \, dx$$

$$= \pi \int_0^1 e^{-x^2} 2x\,dx \ . \quad \text{Substitution } u = x^2$$
$$du = 2x\,dx$$

$$= \pi \int_0^1 e^{-u} du = \pi \left(-e^{-u}\right)\Big|_0^1 =$$

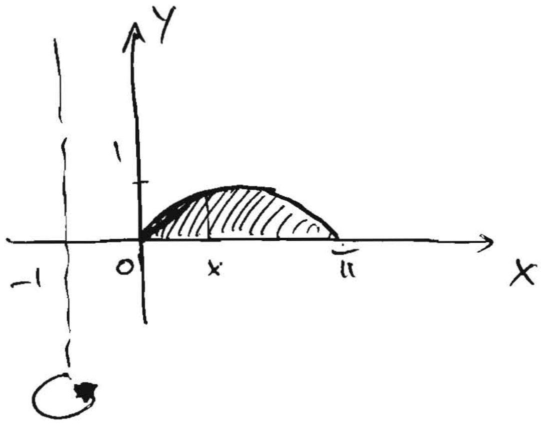
$$= \boxed{\pi\left(1 - e^{-1}\right)}$$

- **Problem 4**
Set up, but DO NOT evaluate, an integral representing the volume of the solid obtained by rotating the region bounded by

$$y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi, \qquad \text{about the line } x = -1$$

Using the method of cyl. shells

$$\text{Volume} = \int_0^\pi 2\pi \underbrace{(x+1)}_{\text{shell radius}} \underbrace{\sin x}_{\text{shell height}} \, dx$$



- **Problem 5**

Find the length of the curve

$$y = 1 + 6x^{3/2}, \quad 0 \le x \le 1$$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \frac{dy}{dx} = 6 \cdot \frac{3}{2} x^{1/2} = 9x^{1/2}$$

$$= \int_0^1 \sqrt{1 + 81x}\, dx$$

$$= \frac{2}{3}(1 + 81x)^{3/2} \cdot \frac{1}{81} \Big|_0^1 = \frac{2}{243}\left(82^{3/2} - 1\right).$$

- **Problem 6**

Set up, but DO NOT evaluate, an integral for the length of the curve

$$y = \ln(\cos x), \quad 0 \le x \le \frac{\pi}{4}$$

$$L = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) = -\tan x$$

$$= \int_0^{\pi/4} \sqrt{1 + (-\tan x)^2}\, dx$$

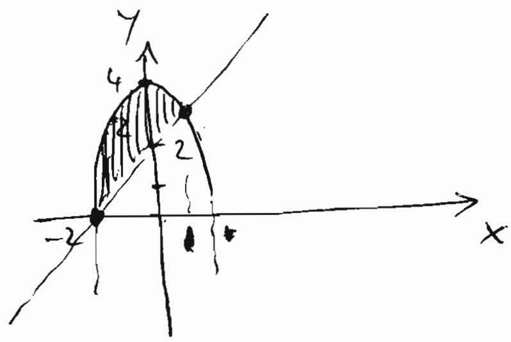
$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/4} \sec x\, dx.$$

- **Problem 7**
  Find the centroid of the region bounded by the curves

$$y = 4 - x^2, \quad y = x + 2.$$



Area : $A = \int_{-2}^{1} [(4-x^2)-(x+2)]\,dx$

$$= \int_{-2}^{1} (2-x-x^2)\,dx$$

$$= \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)$$

$$= 8 - \frac{1}{2} - \frac{9}{3} = \boxed{\frac{9}{2}}$$

Intersection points:

$4 - x^2 = x + 2$

$\Rightarrow x^2 + x - 2 = 0 \Rightarrow$

$\Rightarrow (x-1)(x+2) = 0 \Rightarrow$

$\Rightarrow x = 1$ and $x = -2$

continued below.

- **Problem 8**
  Solve the differential equation

$$y' = \frac{(1+y^2)\cos x}{y} \Rightarrow \frac{dy}{dx} = \frac{1+y^2}{y}\cos x$$

$$\Rightarrow \frac{y}{1+y^2}\,dy = \cos x\,dx \Rightarrow \int \frac{y}{1+y^2}\,dy = \int \cos x\,dx$$

$$\Rightarrow \frac{1}{2}\ln(1+y^2) = \sin x + c \Rightarrow \ln(1+y^2) = 2\sin x + c$$

$$\Rightarrow 1 + y^2 = ce^{2\sin x} \Rightarrow y^2 = ce^{2\sin x} - 1 \Rightarrow \boxed{y = \pm\sqrt{ce^{2\sin x} - 1}}$$

$$\bar{x} = \frac{1}{A}\int_{-2}^{1} x\,[(4-x^2)-(x+2)]\,dx = \frac{1}{A}\int_{-2}^{1} x\,(2-x-x^2)\,dx$$

$$= \frac{1}{A}\int_{-2}^{1}(2x - x^2 - x^3)\,dx = \frac{2}{9}\left(x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_{-2}^{1} = \frac{2}{9}\left(1 - \frac{1}{3} - \frac{1}{4} - \left(4 + \frac{8}{3} - 4\right)\right)$$

$$= \frac{2}{9}\left(-2 - \frac{1}{4}\right) = \frac{2}{9}\left(-\frac{9}{4}\right) = \boxed{-\frac{1}{2}}$$

$$\bar{y} = \frac{1}{A}\int_{-2}^{1}\frac{1}{2}\left[(4-x^2)^2 - (x+2)^2\right]dx = \frac{1}{2A}\int_{-2}^{1}(16 - 8x^2 + x^4 - x^2 - 4x - 4)\,dx$$

$$= \frac{1}{2A}\int_{-2}^{1}(12 - 9x^2 - 4x + x^4)\,dx = \frac{1}{9}\left(12x - 2x^2 - 3x^3 + \frac{x^5}{5}\right)\Big|_{-2}^{1}$$

$$= \frac{1}{9}\left[\left(12 - 2 - 3 + \frac{1}{5}\right) - \left(-24 - 8 + 24 - \frac{32}{5}\right)\right] = \frac{1}{9}\left(15 + \frac{33}{5}\right) = \frac{1}{9}\cdot\frac{108}{5} = \boxed{\frac{12}{5}}$$