5.6 Solutions

5.1 Suppose \( T \in \mathcal{L}(\mathbb{V}) \). Prove that if \( U_1, \ldots, U_M \) are subspaces of \( \mathbb{V} \) invariant under \( T \), then \( U_1 + \cdots + U_M \) is invariant under \( T \).

Suppose \( v = u_1 + \cdots + u_N \in U_1 + \cdots + U_M \), with \( u_n \in U_n \). Then

\[
T v = Tu_1 + \cdots + Tu_N.
\]

Since each \( U_n \) is invariant under \( T \), \( Tu_n \in U_n \), so \( T v \in U_1 + \cdots + U_M \).

5.2 Suppose \( T \in \mathcal{L}(\mathbb{V}) \). Prove that the intersection of any collection of subspaces of \( \mathbb{V} \) invariant under \( T \) is invariant under \( T \).

Suppose we have a set of subspaces \( \{U_r\} \), with each \( U_r \) invariant under \( T \). Let \( v \in \cap_r U_r \). Then \( Tv \in U_r \) for each \( r \), and so \( \cap_r U_r \) is invariant under \( T \).

5.3 Prove or give a counterexample: if \( U \) is a subspace of \( \mathbb{V} \) that is invariant under every operator on \( \mathbb{V} \), then \( U = \{0\} \) or \( U = \mathbb{V} \).

We'll prove the contrapositive: if \( U \) is a subspace of \( \mathbb{V} \) and \( U \neq \{0\} \) and \( U \neq \mathbb{V} \), then there is an operator \( T \) on \( \mathbb{V} \) such that \( U \) is not invariant under \( T \).

Let \( (u_1, \ldots, u_M) \) be a basis for \( U \), which we extend to a basis

\[
(u_1, \ldots, u_M, v_1, \ldots, v_N)
\]

of \( \mathbb{V} \). The assumption \( U \neq \{0\} \) and \( U \neq \mathbb{V} \) means that \( M \geq 1 \) and \( N \geq 1 \). Define a linear map \( T \) by

\[
Tu_1 = v_1, \quad Tu_n = 0, \quad n > 1.
\]

Since \( v_1 \notin U \), the subspace \( U \) is not invariant under the operator \( T \).

5.4 Suppose \( S, T \in \mathcal{L}(\mathbb{V}) \) are such that \( ST = TS \). Prove that \( \text{null}(T - \lambda I) \) is invariant under \( S \) for every \( \lambda \in \mathbb{F} \).

Suppose that \( Tv = \lambda v \). Then \( Sv \) satisfies

\[
T(Sv) = S(Tv) = S\lambda v = \lambda(Sv).
\]

Thus if \( v \) is an eigenvector of \( T \) with eigenvalue \( \lambda \), so is \( Sv \).

5.5 Define \( T \in \mathcal{L}(\mathbb{F}^2) \) by

\[
T(w, z) = (z, w).
\]
Find all eigenvalues and eigenvectors of $T$. Suppose $(w, z) \neq (0, 0)$ and

$$T(w, z) = (z, w) = \lambda(w, z).$$

Then

$$z = \lambda w, \quad w = \lambda z.$$

Of course this leads to

$$w = \lambda z = \lambda^2 w, \quad z = \lambda w = \lambda^2 z.$$

Since $w \neq 0$ or $z \neq 0$, we see that $\lambda^2 = 1$, so $\lambda = \pm 1$.

A basis of eigenvectors is

$$(w_1, z_1) = (1, 1), \quad (w_2, z_2) = (-1, 1),$$

and they have eigenvalues 1 and $-1$ respectively.

5.6 Define $T \in \mathcal{L}(\mathbb{F}^3)$ by

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3).$$

Find all eigenvalues and eigenvectors of $T$. Suppose $(z_1, z_2, z_3) \neq (0, 0, 0)$ and

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3).$$

If $\lambda = 0$ then $z_2 = z_3 = 0$, and one checks easily that

$$v_1 = (1, 0, 0)$$

is an eigenvector with eigenvalue 0. If $\lambda \neq 0$ then $z_2 = 0$,

$$2z_2 = \lambda z_1 = 0, \quad 5z_3 = \lambda z_3,$$

so $z_1 = 0$ and $\lambda = 5$. The eigenvector for $\lambda = 5$ is

$$v_2 = (0, 0, 1).$$

These are the only eigenvalues, and each eigenspace is one dimensional.

5.7 Suppose $N$ is a positive integer and $T \in \mathcal{L}(\mathbb{F}^N)$ is defined by

$$T(x_1, \ldots, x_N) = (x_1 + \cdots + x_N, \ldots, x_1 + \cdots + x_N).$$

Find all eigenvalues and eigenvectors of $T$. 

First, any vector of the form
\[ \mathbf{v}_1 = (\alpha, \ldots, \alpha), \quad \alpha \in \mathbb{F}, \]
is an eigenvector with eigenvalue \( N \). If \( \mathbf{v}_2 \) is any vector
\[ \mathbf{v}_2 = (x_1, \ldots, x_N), \quad \sum_n x_n = 0, \]
then \( \mathbf{v}_2 \) is an eigenvector with eigenvalue 0.

Here are \( N \) independent eigenvectors:
\[ \mathbf{v}_1 = (1, 1, \ldots, 1), \]
and
\[ \mathbf{v}_n = (1, 0, \ldots, 0) - E_n, \quad n \geq 2, \]
where \( E_n \) denotes the \( n \)-th standard basis vector.

5.10 \( T \in \mathcal{L}(\mathbb{V}) \) is invertible and \( \lambda \in \mathbb{F} \setminus \{0\} \). Prove that \( \lambda \) is an eigenvalue of \( T \) if and only if \( 1/\lambda \) is an eigenvalue of \( T^{-1} \).

Suppose \( \mathbf{v} \neq 0 \) and \( T \mathbf{v} = \lambda \mathbf{v} \). Then
\[ \mathbf{v} = T^{-1}T \mathbf{v} = \lambda T^{-1} \mathbf{v} , \]
or
\[ T^{-1} \mathbf{v} = \frac{1}{\lambda} \mathbf{v} , \]
and the other direction is similar.

5.11 Suppose \( S, T \in \mathcal{L}(\mathbb{V}) \). Prove that \( ST \) and \( TS \) have the same eigenvalues.

Suppose \( \mathbf{v} \neq 0 \) and \( ST \mathbf{v} = \lambda \mathbf{v} \). Multiply by \( T \) to get
\[ TST \mathbf{v} = \lambda T \mathbf{v} . \]

Thus if \( T \mathbf{v} \neq 0 \) then \( \lambda \) is also an eigenvalue of \( TS \), with nonzero eigenvector \( T \mathbf{v} \).

On the other hand, if \( T \mathbf{v} = 0 \), then \( \lambda = 0 \) is an eigenvalue of \( ST \). But if \( T \) is not invertible, then \( \text{range} T \subseteq \text{range} TS \) is not equal to \( \mathbb{V} \), so \( TS \) has a nontrivial null space, hence 0 is an eigenvalue of \( TS \).

5.12 Suppose \( T \in \mathcal{L}(\mathbb{V}) \) is such that every vector in \( \mathbb{V} \) is an eigenvector of \( T \). Prove that \( T \) is a scalar multiple of the identity operator.
5.6. SOLUTIONS

Pick a basis \((v_1, \ldots, v_N)\) for \(V\). By assumption,
\[ Tv_n = \lambda_n v_n. \]

Pick any two distinct indices, \(m, n\). We also have
\[ T(v_m + v_n) = \lambda(v_m + v_n) = \lambda_m v_m + \lambda_n v_n. \]

Write this as
\[ 0 = (\lambda - \lambda_m)v_m + (\lambda - \lambda_n)v_n. \]

Since \(v_m\) and \(v_n\) are independent,
\[ \lambda = \lambda_m = \lambda_n, \]

and all the \(\lambda_n\) are equal.

5.14 Suppose \(S, T \in \mathcal{L}(V)\) and \(S\) is invertible. Prove that if \(p \in \mathcal{P}(F)\) is a polynomial, then
\[ p(STS^{-1}) = Sp(T)S^{-1}. \]

First let’s show that for positive integers \(n\),
\[ (STS^{-1})^n = ST^n S^{-1}. \]

We may do this by induction, with nothing to show if \(n = 1\). Assume it’s true for \(n = k\), and consider
\[ (STS^{-1})^{k+1} = (STS^{-1})^k(STS^{-1}) = ST^k S^{-1} S T S^{-1} = ST^{k+1} S^{-1}. \]

Now suppose
\[ p(z) = a_n z^n + \cdots + a_1 z + a_0. \]

Then
\[ p(STS^{-1}) = \sum a_n (STS^{-1})^n = \sum a_n ST^n S^{-1} \]
\[ = S(\sum a_n T^n) S^{-1} = Sp(T)S^{-1}. \]

5.15 Suppose \(F = \mathbb{C}\), \(T \in \mathcal{L}(V)\), \(p \in \mathcal{P}(\mathbb{C})\), and \(\alpha \in \mathbb{C}\). Prove that \(\alpha\) is an eigenvalue of \(p(T)\) if and only if \(\alpha = p(\lambda)\) for some eigenvalue \(\lambda\) of \(T\).

Suppose first that \(v \neq 0\) is an eigenvector of \(T\) with eigenvalue \(\lambda\). That is
\[ Tv = \lambda v. \]
Then for positive integers \( n \),
\[
T^n v = T^{n-1} \lambda v = \cdots = \lambda^n v,
\]
and so
\[
p(T)v = p(\lambda)v.
\]
That is \( \alpha = p(\lambda) \) is an eigenvalue of \( p(T) \) if \( \lambda \) is an eigenvalue of \( T \).
Suppose now that \( \alpha \) is an eigenvalue of \( p(T) \), so there is a \( v \neq 0 \) with
\[
p(T)v = \alpha v,
\]
or
\[
(p(T) - \alpha I)v = 0.
\]
Since \( \mathbb{F} = \mathbb{C} \), we may factor the polynomial \( p(T) - \alpha I \) into linear factors
\[
0 = (p(T) - \alpha I)v = \prod_n (T - \lambda_n I)v.
\]
At least one of the factors is not invertible, so at least one of the \( \lambda_n \), say \( \lambda_1 \), is an eigenvalue of \( T \). Let \( w \neq 0 \) be a eigenvector for \( T \) with eigenvalue \( \lambda_1 \). Then
\[
0 = (T - \lambda_1 I)w = (p(T) - \alpha I)w,
\]
so \( w \) is an eigenvector for \( p(T) \) with eigenvalue \( \alpha \). But by the first part of the argument,
\[
p(T)w = p(\lambda_1)w = \alpha w,
\]
and \( \alpha = p(\lambda_1) \).

5.16 Show that the result in the previous exercise does not hold if \( \mathbb{C} \) is replaced with \( \mathbb{R} \).
Take \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) given by
\[
T(x, y) = (-y, x).
\]
We’ve seen previously that \( T \) has no real eigenvalues. On the other hand,
\[
T^2(x, y) = (-x, -y) = -1 \cdot (x, y).
\]

5.17 Suppose \( \mathbb{V} \) is a complex vector space and \( T \in \mathcal{L}(\mathbb{V}) \). Prove that \( T \) has an invariant subspace of dimension \( j \) for each \( j = 1, \ldots, \dim \mathbb{V} \).
Let $(v_1, \ldots, v_N)$ be a basis with respect to which $T$ has an upper triangular matrix. Then by Proposition 5.12

$$T : \text{span}(v_1, \ldots, v_j) \rightarrow \text{span}(v_1, \ldots, v_j).$$

5.18 Give an example of an operator whose matrix with respect to some basis contains only 0’s on the diagonal, but the operator is invertible.

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

5.19 Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

We can use the idea of problem 5.7, taking

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

If $v = [1, -1]$, then $Tv = 0$. 