

Show All Your Work

1. (15 pts) Suppose the function $f(x)$ is continuous on $[0, 1]$, and there is a root r satisfying $0 < r < 1$, with

$$f(x) < 0, \quad x < r,$$

$$f(x) > 0, \quad x > r.$$

Describe how to use the bisection method to find r . How many iterations should you use to make sure that the error in estimating r is smaller than 10^{-6} ? Explain.

Define sequences $\{a_n\}$ and $\{b_n\}$ with $a_0 = 0$ and $b_0 = 1$. For $n = 1, 2, 3, \dots$ proceed as follows. First, let $c = (a_{n-1} + b_{n-1})/2$. If $f(c) = 0$ stop; take $r = c$. If $f(c) < 0$ let $a_n = c$, $b_n = b_{n-1}$. If $f(c) > 0$ let $a_n = a_{n-1}$, $b_n = c$.

Continue until $|b_N - a_N|$ is sufficiently small, then estimate r with either $r = b_N$ or $r = a_N$.

Since the length of the interval $[a_n, b_n]$ containing r is halved each step, and $b_0 - a_0 = 1$, the maximum error after N steps is 2^{-N} . Thus we want

$$2^{-N} \leq 10^{-6}, \quad N \log_{10}(2) \geq 6.$$

It will suffice to take $N \geq 24$.

2. (15 pts) A manufacturing firm has a profit model which is a function $f(p_1, \dots, p_{25})$ of the production levels of 25 different products. The firm believes that plausible production levels for each product range from 0 to 10^6 units per year. Management wants an annual production plan for each product in the form of $10^4 N$ units, where the integer N ranges from 0 to 100.

Would exhaustive search be a practical approach for optimizing the profit in this problem? Explain.

Why might random search appear more attractive for optimizing the profit? Briefly describe how you would go about doing such a random search.

In this case an exhaustive search is highly impractical. Suppose we want to test each possible production plan. This would involve independently assigning 25 variables the values $0, \dots, 100$. The number of such

assignments is $(101)^{25}$, a number of function evaluations far to great for a computer to check.

The number of steps in a random search can be controlled. Say you are willing to perform M function evaluations. For each one, have the computer generate 25 (pseudo-) random integers uniformly distributed from 0 to 100. Evaluate the profit of each production plan, saving the values giving the most profitable plan.

3. (20 pts) a) Write down Newton's method for estimating solutions of

$$g_1(x, y) = 0,$$

$$g_2(x, y) = 0.$$

b) Suppose we want to minimize

$$f(x, y) = x^2 + y^2 - 2x - y + 1.25.$$

How would you do this with Newton's method? Given the initial guess $x_0 = 0$, $y_0 = 0$, calculate x_1 , y_1 .

a) Newton's method starts with an initial guess x_0, y_0 . Let

$$A_n = \begin{pmatrix} \partial g_1 / \partial x & \partial g_1 / \partial y \\ \partial g_2 / \partial x & \partial g_2 / \partial y \end{pmatrix} (x_n, y_n).$$

The sequence of Newton's method estimates is given by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - A_n^{-1} \begin{pmatrix} g_1(x_n, y_n) \\ g_2(x_n, y_n) \end{pmatrix}.$$

b) To minimize f , look for solutions of $\nabla f = (0, 0)$. We have

$$\nabla f = \begin{pmatrix} g_1(x, y) \\ g_2(x, y) \end{pmatrix} = \begin{pmatrix} \partial f / \partial x \\ \partial f / \partial y \end{pmatrix} = \begin{pmatrix} 2x - 2 \\ 2y - 1 \end{pmatrix}.$$

In this case

$$A_n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

With the initial guess $x_0 = 0$, $y_0 = 0$, we find

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}.$$

(By the way, this is the exact solution of the problem.)

4. (20 pts) Consider the following inequality program: maximize $P = 10x_1 + 200x_2 + 35x_3$, subject to

$$x_1 + 4x_2 + 3x_3 \leq 100,$$

$$x_1 - x_2 + x_3 \leq 50,$$

$$x_1 - x_2 - x_3 \leq 25,$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

(a) By introducing slack variables x_4, x_5, x_6 , rewrite this problem as an equality linear program.

Maximize $P = 10x_1 + 200x_2 + 35x_3$, subject to

$$x_1 + 4x_2 + 3x_3 + x_4 = 100,$$

$$x_1 - x_2 + x_3 + x_5 = 50,$$

$$x_1 - x_2 - x_3 + x_6 = 25,$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

(b) Show that the feasible set F for the equality program is bounded.

We add the first and second equations to the third, which does not change the solutions but gives the constraint equations as

$$x_1 + 4x_2 + 3x_3 + x_4 = 100,$$

$$x_1 - x_2 + x_3 + x_5 = 50,$$

$$3x_1 + 2x_2 + 3x_3 + x_4 + x_5 + x_6 = 175,$$

Since the last equation has all coefficients positive, the feasible set F for the equality program is bounded.

5. (15 pts) Solve the following inequality linear program. You do NOT have to convert this to an equality program. Maximize $P = 3x + 2y$, subject to

$$x + 2y \leq 6, \quad y \leq 2, \quad x \leq 4, \quad x, y \geq 0.$$

The maximum must occur at one of the extreme points, which are

$$(0, 0), (0, 2), (4, 0), (2, 2), (4, 1).$$

Evaluation of P at each of these points shows that the maximum is $P = 14$ at $x = 4, y = 1$.

6. (15 pts) Suppose $\Omega \subset R^n$.

a) Define what it means for Ω to be convex.

b) Show that the intersection of two convex sets Ω_1 and Ω_2 is convex.

c) Define what it means for $X \in \Omega$ to be an extreme point.

a) $\Omega \subset R^n$ is convex if whenever $X, Y \in \Omega$, then

$$tX + (1 - t)Y \in \Omega, \quad 0 \leq t \leq 1.$$

b) Suppose $X, Y \in \Omega_1$ and $X, Y \in \Omega_2$. Since Ω_1 is convex,

$$tX + (1 - t)Y \in \Omega_1, \quad 0 \leq t \leq 1.$$

Similarly for Ω_2 . Thus

$$tX + (1 - t)Y \in \Omega_1 \cap \Omega_2, \quad 0 \leq t \leq 1.$$

c) $X \in \Omega$ is an extreme point if whenever

$$X = tY_1 + (1 - t)Y_2, \quad Y_1, Y_2 \in \Omega, \quad 0 < t < 1,$$

then $Y_1 = Y_2 = X$.