Test 2   Math 448/548   Professor Carlson

Show All Your Work

1. *(30 pts) Growth for a single population is often modeled with one of two equations:*

$$\frac{dp}{dt} = rp, \quad r > 0, \tag{A}$$

$$\frac{dp}{dt} = rp(1 - p/K), \quad r > 0,\ K > 0. \tag{B}$$

a) *Sketch typical solution curves for these two models.*

b) *Interpret the parameters r and K.*

The parameter r is the intrinsic growth parameter. It characterizes the maximum rate of growth in the absence of environmental constraints acting to limit growth. The parameter K is the environmental carrying capacity or maximum sustainable population. Positive populations smaller than K grow, have K as a limit, but remain less than K. Positive populations larger than K decay, have K as a limit, but remain greater than K.

c) *Give some reasonable conditions under which the simpler model (A) would be acceptable. For what kinds of information would you need to consider model (B)?*

The simpler model (A) is generally acceptable if population growth is not significantly impacted by environmental constraints. This is the case if relatively small numbers of organisms are introduced to a new environment or a population within an environment experiences a dramatic reduction in size due to some factor outside the model. (Example - bacteria start growing in a cooled pot of soup.)

The more elaborate model (B) is appropriate if a single population is large relative to the carrying capacity, near equilibrium ($p \simeq K$) or is being modeled for a sufficiently long time that it will approach such an equilibrium.

2. (30 pts) *Two species in competition are modeled by the dynamical system*

$$\frac{dx}{dt} = x(1 - 10^{-4}x) - 10^{-6}xy = f(x,y),$$

$$\frac{dy}{dt} = y(1 - .5 * 10^{-4}y) - 10^{-6}xy = g(x,y).$$

a) *Find the equilibrium solution with both populations positive.*
Solve $x' = y' = 0$ to get

$$1 - 10^{-4}x - 10^{-6}y = 0, \quad 1 - .5 * 10^{-4}y - 10^{-6}x = 0$$

or

$$x = \frac{9800}{1 - 2 * 10^{-4}} \simeq 10^4, \quad y = \frac{1 - 10^{-2}}{.5 * 10^{-4} - 10^{-8}} \simeq 2 * 10^4.$$

b) Sketch the vector field. Is the equilibrium solution of part a) stable?

c) *Analyze the stability of the equilibrium solution of part a) using the eigenvalue method. You may approximate the equilibrium point by*

$$x \simeq 10^4, \quad y \simeq 2 * 10^4.$$

The stability is determined by the eigenvalues of the matrix

$$\begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}$$

at the equilibrium point, which we take to be $(10^4, 2 * 10^4)$. We have

$$\partial f/\partial x = 1 - 2 * 10^{-4}x - 10^{-6}y = -1.02,$$

$$\partial f/\partial y = -10^{-2},$$

$$\partial g/\partial x = -2 * 10^{-2},$$

$$\partial g/\partial y = 1 - 2 * .5 * 10^{-4}y - 10^{-6}x = -1.01.$$

Thus at the equilibrium point

$$\begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} = \begin{pmatrix} -1.02 & -10^{-2} \\ -2 * 10^{-2} & -1.01 \end{pmatrix}.$$

The eigenvalues, which you can calculate exactly, are approximately $-1, -1$, so the equilibrium point is stable.

3. (30 pts) *Consider the following inequality program: maximize* $P = 100x_1 + 200x_2 + 300x_3$, *subject to*

$$2x_1 + 3x_2 + 4x_3 \leq 100,$$

$$x_1 + -x_2 + x_3 \leq 50,$$

$$x_1 - x_2 - x_3 \leq 25,$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

(a) *By introducing slack variables x_4, x_5, x_6, rewrite this problem as an equality linear program.*

Maximize $P = 100x_1 + 200x_2 + 300x_3$, subject to

$$2x_1 + 3x_2 + 4x_3 + x_4 = 100,$$

$$x_1 + -x_2 + x_3 + x_5 = 50,$$

$$x_1 - x_2 - x_3 + x_6 = 25,$$

$$x_1 \geq 0, \dots, x_6 \geq 0.$$

(b) *Show that the feasible set F for the equality program is bounded.*

Since $x_n \geq 0$ the first equation bounds $x_1, \dots, x_4$. For instance

$$x_1 \leq 100, \dots, x_4 \leq 100.$$

The second equation then gives

$$|x_5| \leq |50 - x_1 + x_2 - x_3| \leq 50 + |x_1| + |x_2| + |x_3| \leq 350,$$

and similarly the third equation bounds x_6.

c) *Solve the following inequality linear program. You do NOT have to convert this to an equality program. Maximize $P = x + 2y$, subject to*

$$x + y \leq 10, \quad y \leq 8, \quad x, y \geq 0.$$

The maximum must occur at one of the extreme points of the feasible set. The extreme points (draw a picture) are at

$$(x, y) = (0, 0), \quad (x, y) = (10, 0), \quad (x, y) = (2, 8), \quad (x, y) = (0, 8).$$

Checking the values of $P(x, y)$ at the extreme points, we find the maximum value $P = 18$ at $(x, y) = (2, 8)$.

4. (10 pts)*A manufacturer's operations research department has developed a model for profit which is a complicated function of the production levels of* 20 *different products. They believe that plausible production levels for each product range from* 0 *to* 10^6 *units per year. Management wants a production plan for each product in the form of* $10^4 N$ *units, where* N *ranges from* 0 *to* 100.

Why would random search appear to be an appropriate technique for optimization? Briefly describe how you would go about doing such a random search.

Since the profit model is complicated and otherwise unstated, our main alternatives are systematic search or random search over the possible production levels. Since there are 20 products p_i, with production levels N_i which are integers in the range $0 \leq N_i \leq 100$, the number of different production allocations is $(101)^{20} \simeq 10^{40}$. This number is much too large for a computer to check each case. Thus a randomized search is the only viable option.

Have a random number generator randomly select vectors of 20 integers $(N_1, N_2, \ldots, N_{20})$ with $0 \leq N_i \leq 100$. Report the highest computed profit. Notice that you only have to consider the biggest profit yet computed (one number) and the profit value for the current vector $(N_1, N_2, \ldots, N_{20})$, so very little data is ever saved.