Test 1 Math 448/548 Professor Carlson

Show all your work

1. (a) (10 pts) *Find the minimum of the function*

$$f(x,y) = 2x^2 - 2xy + y^2 - 4x + 4.$$

(You may assume the minimum exists.)

Solve the system of equations

$$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y},$$

which is

$$4x - 2y - 4 = 0,$$

$$-2x + 2y = 0.$$

The solution is $x = 2$, $y = 2$.

(b) (15pts) *Use Lagrange multipliers to maximize and minimize the function*

$$f(x,y,z) = (x+y)^2$$

subject to the constraint

$$x^2 + 2y^2 + 4z^2 = 6.$$

We have $\nabla f = (2(x+y), 2(x+y), 0)$ and $\nabla g =)2x, 4y, 8z)$. The equation $\nabla f = \lambda \nabla g$ leads to the system of equations

$$2(x+y) = 2\lambda x,$$

$$2(x+y) = 4\lambda y,$$

$$0 = 8\lambda z,$$

$$x^2 + 2y^2 + 4z^2 = 6.$$

If $\lambda \neq 0$ we have $z = 0$ and $x = 2y$, leading to the solutions $(2,1,0)$ and $(-2,-1,0)$.

Notice that the minimum of f is the value 0 when $x + y = 0$, which is also the condition we obtain with $\lambda = 0$. The value of f is minimized for any z if $x + y = 0$, as long as the constraint is satisfied. If $y = -x$, the constraint becomes

$$3x^2 + 4z^2 = 6.$$

Thus we can choose any x and z values appearing on the ellipse $3x^2 + 4z^2 = 6$, and then take $y = -x$.

Since the set where $x^2 + 2y^2 + 4z^2 = 6$ is closed and bounded, f must achieve a minimum and maximum. We just discussed the minimum, so the maximum must be the initial solutions $(2, 1, 0)$ and $(-2, -1, 0)$, where $f = 9$.

2. (25 pts) *Consider using Newton's method in several variables to maximize*

$$f(x, y) = x + y - 2xy - x^2 - y^2/2$$

as follows.

a) *Compute the gradient*

$$\begin{pmatrix} g_1(x, y) \\ g_2(x, y) \end{pmatrix} = \nabla f(x, y).$$

$$G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \nabla f = \begin{pmatrix} 1 - 2y - 2x \\ 1 - 2x - y \end{pmatrix}.$$

b) *Write down Newton's method in several variables.* Let

$$DG = \begin{pmatrix} \partial g_1/\partial x & \partial g_1/\partial y \\ \partial g_2/\partial x & \partial g_2/\partial y \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -2 & -1 \end{pmatrix}.$$

Pick

$$X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

and then the Newton's method sequence is defined by

$$X_{n+1} = X_n - (DG)^{-1}(X_n)G(X_n).$$

c) *Given the initial guess $x_0 = 0$, $y_0 = 0$, calculate x_1, y_1 with Newton's method.*

In this case

$$(DG)^{-1} = \frac{-1}{2}\begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix},$$

so if $x_0 = y_0 = 0$ we find

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.$$

3. (25 pts) *An oil spill has fouled 200 miles of Pacific shoreline. The responsible oil company has been given* 14 *days to clean the shore, after which the company will be fined* \$10,000/*day. A crew of workers can clean* 5 *miles of beach per week. There is* 1 *local crew available, at a cost of* \$500/*day. Additional crews can be brought in at a cost of* \$18,000 *plus* \$800/*day for each crew.*

(a) *Set up a model for cost and clean up time, then express cost as a function of one variable, the number of additional crews. Without solving the problem, explain briefly how it can be solved so the cost is minimized.*

Let x be the number of additional crews, t be the clean up time in days, and c be the cost. If it takes more than 14 days, then

$$c = 10,000(t - 14) + 500t + x(18,000 + 800t),$$

$$t = \frac{7}{5}\frac{200}{x+1} = \frac{280}{x+1}.$$

We then find

$$c_1(x) = 10,000(\frac{280}{x+1} - 14) + 500\frac{280}{x+1} + x(18,000 + 800\frac{280}{x+1}).$$

If it takes less than 14 days, then

$$c_2 = 500t + x(18,000 + 800t) = 500\frac{280}{x+1} + x(18,000 + 800\frac{280}{x+1}).$$

To minimize the cost, evaluate cost at those x where

$$c'(x) = 0,$$

and at $t = 280/(x+1) = 14$.

(b) *Suppose the anticipated cost with the original model is* \$500,000, *with a clean up time of* 24 *days. Explain how you would calculate the sensitivity of the total cost with respect to the amount of fine per day. You do not have to carry out the calculation.*

Introduce a new variable f equal to the amount of fine per day. Find minimum cost $C(f)$ as a function of f. The sensitivity is

$$\frac{dC}{df}\frac{f}{C}$$

evaluated at $f = \$10,000$ and $C = \$500,000$.

(c) *Suppose the sensitivity of the clean up time with respect to the amount of fine per day is* -0.5. *Estimate the clean up time if the fine is raised to* \$15,000/*day.*

This is a 50% increase in fine per day. Since the sensitivity is -0.5, the clean up time should decrease by about 25%, giving a total clean up time of 18 days.

4. (25 pts) *Suppose the population* $p(t)$ *of tuna is modelled by the equation*

$$\frac{dp}{dt} = rp(1 - p/K),$$

where r *is the intrinsic growth rate, and* K *is the maximum sustainable population.*

(a) *For which value* p_{max} *is the rate of growth maximized? What is the rate* R_{max} *of population growth at* p_{max}*? What is the sensitivity of* R_{max} *with respect to the maximum sustainable population?*

If

$$R(p) = rp(1 - p/K),$$

differentiation shows that the maximum occurs at $p_{max} = K/2$.

The corresponding rate is

$$R_{max} = rK/4.$$

The sensitivity is

$$S = \frac{dR_{max}}{dK}\frac{K}{R_{max}} = \frac{r}{4}\frac{4K}{rK} = 1.$$

(b) *Suppose a disease reduces the population growth, which is now modelled as*

$$\frac{dp}{dt} = rp(1 - p/K) - rK/8.$$

What constant levels of population are predicted? For what population values is the population increasing?

Constant levels of population have $dp/dt = 0$, or

$$rp(1 - p/K) - rK/8 = 0.$$

Solving for p we find

$$p_{\pm} = K(1 \pm 1/\sqrt{2})/2.$$

The population is increasing when $rp(1 - p/K) - rK/8 > 0$. Since the leading coefficient of $rp(1 - p/K) - rK/8$ is negative, the population is increasing when $p_- < p < p_+$.