Math 448/548 Professor Carlson

Test 1 Solutions

1. (12 pts) *Find the minimum of the function*

$$f(x,y) = x^2 + y^2 - 6x - 4y + xy.$$

(You may assume the minimum exists.)

If the minimum exists it must occur when

$$\partial f/\partial x = \partial f/\partial y = 0.$$

This gives two equations

$$2x + y = 6, \quad 2y + x = 4.$$

The solution is

$$x = 8/3, \quad y = 2/3.$$

2. (15 pts) *The average pig has weight growth described by the function $w(t)$. A farmer has a pig whose current weight, at $t = 0$, is 200 pounds. The pig costs 45 cents per day to keep. The market price for pigs is 65 cents per pound, but is falling 1 cent per day.*

Consider two models for the weight of the pig in pounds. In case (i) we have

$$w(t) = 200 + 5t.$$

In case (ii) the weight is described by the equation

$$\frac{dw}{dt} = 1.25 \times 10^{-4} w(400 - w).$$

(a) Using $P = R - C$, that is profit = revenue - cost, write down the profit for a general weight function $w(t)$.

(b) By setting $dP/dt = 0$ and using the case (ii) model, derive an equation in t and w for the optimal profit in case (ii). (Do not try to solve this equation.)

(c) Suppose you know that the growth rate in model (ii) is never more than 5 pounds per day, which is the constant rate of model (i). Will the

maximum profit predicted by model (ii) be greater or less than that predicted by model (i)? Explain your reasoning.

(a) Profit is

$$P = R - C = (.65 - .01t)w(t) - .45t.$$

(b) Differentiation gives

$$dP/dt = (-.01)w + (.65 - .01t)w'(t) - .45.$$

Setting the derivative equal to 0 and using the differential equation from model (ii) we find

$$0 = (-.01)w + (.65 - .01t)[1.25 \times 10^{-4} w(400 - w)] - .45.$$

(c) Since the pig weighs 200 pounds when $t = 0$, and the growth rate in model (ii) is never more than 5 pounds per day, which is the constant rate of model (i), the weight $w_1(t)$ predicted by model (i) is always greater than or equal to the weight $w_2(t)$ predicted by model (ii), for $t \geq 0$. Thus at any time $0 \leq t \leq 65$ we find $P_1(t) \geq P_2(t)$; the profit is less for model (ii).

3. (25pts) Find the point on the curve

$$(x - 4)^2 + (y - 2)^2 + (x - 4)(y - 2) = 1$$

which is closest to the y-axis as follows.

a) Set up the problem of minimizing the function $f(x, y) = x^2$ subject to the given constraint as a Lagrange multiplier problem.

b) Solve the Lagrange multiplier problem.

a) In our standard notation the objective function is $f(x, y) = x^2$ and the constraint function is $g(x, y) = (x-4)^2 + (y-2)^2 + (x-4)(y-2)$. Look for points (x, y) where

$$\nabla f = \lambda \nabla g.$$

This gives the equations

$$2x = \lambda(2x + y - 10)$$

$$0 = \lambda(2y + x - 8).$$

The second equation means that either $\lambda = 0$ or $2y + x - 8 = 0$. In case $\lambda = 0$ we find $x = 0$ and then that there are no real solutions (x, y) to the equation $(x-4)^2 + (y-2)^2 + (x-4)(y-2) = 1$ with $x = 0$.

With $2y + x - 8 = 0$ we may solve for x and write the constraint as

$$(4 - 2y)^2 + (y - 2)^2 + (4 - 2y)(y - 2) = 1$$

or

$$3(y - 2)^2 = 1.$$

We get candidate solutions

$$y = 2 + 1/\sqrt{3}, x = 4 - 2/\sqrt{3}, \quad y = 2 + -1/\sqrt{3}, x = 4 + 2/\sqrt{3}.$$

The function x^2 is minimized by the first candidate.

4. (25 pts) *The growth rate of a whale population $x(t)$ is modelled by the equation*

$$\frac{dx}{dt} = rx(1 - x/K),$$

where r is the intrinsic growth rate, and K is the maximum sustainable population.

(a) For which value x_{max} is the rate of growth maximized? What is the rate R_{max} of population growth at x_{max}?

(b) The owners of the whaling fleet argue that they should be able to harvest R_{max} whales per year since the current population is greater than $K/2$, and if it drops to $K/2$ the rate of reproduction will replace the harvested whales. What will happen if the population approaches $K/2$, the fishing harvest is sustained at R_{max}, and disease or accidents cause the population to drop below $K/2$ at some point? Explain.

(c) Suppose the government mandates a harvest rate of $.75R_{max}$, so the population growth is modelled by

$$\frac{dx}{dt} = rx(1 - x/K) - .75R_{max}.$$

What constant levels of population are predicted by this model? Will this successfully address the problem raised in part (b)? Explain.

(a) Set the x derivative of $rx(1 - x/K)$ equal to 0 to maximize. You find $x = K/2$ and $R_{max} = rK/4$.

(b) If there is ever a time t_0 when $x(t_0) < K/2$ then the growth rate will be less than the harvest rate, with an increasing gap in the future. The population will be driven to 0.

(c) A constant level of population has $dx/dt = 0$. Solving for x we find

$$x = \frac{K}{2}(1 \pm .5) = 3K/4, K/4.$$

By checking the sign of the derivative we see that $dx/dt > 0$ for $K/4 < x < 3K/4$. Thus if an accident drives the population slightly below $3K/4$, the population will subsequently increase. This will successfully address the problem raised in part (b).

5. (25 pts) A local newspaper with 80,000 subscribers is considering a price rise. Currently the price is \$1.50 *per week, and it is estimated that the paper would lose 5,000 subscribers for each price increase of ten cents per week.*

a) Find the subscription price that maximizes income.

b) Calculate the sensitivity of this optimum price to the assumption of losing 5,000 subscribers for each price increase of ten cents per week.

a) I prefer to describe income R as a function of the price increase p in cents. Then

$$R(p) = (80,000 - 500p)(150 + p).$$

Look for the critical points $R'(p) = 0$,

$$R'(p) = -500(150 + p) + (80,000 - 500p) = 0, \quad p = 5.$$

Thus the optimal price is \$1.55. (Since R is a quadratic polynomial, a look at the leading term shows it has a max.)

b) Introduce the new parameter a so that

$$R(p, a) = (80,000 - \frac{a}{10}p)(150 + p).$$

Solving

$$\frac{dR}{dp} = 0$$

we get

$$p = \frac{40,000}{a} - 75.$$

This is the optimal price increase, so the optimal price is

$$P = 150 + \frac{40,000}{a} - 75 = \frac{40,000}{a} + 75.$$

The sensitivity is

$$S(P, a) = \frac{dP}{da}\frac{a}{P} = -\frac{40,000}{a^2}\frac{a}{P} = -\frac{40,000}{aP}.$$

Plugging in $P = 150$ and $a = 5000$ gives

$$S(P, a) = -5.3.$$