2.5 The Simplex Algorithm

Theorem 2.12 shows that the linear program can be solved by checking only extreme points, and Theorem 2.13 shows that there is only a finite number of extreme points. As a practical matter, however, it is usually impossible to check all extreme points. Suppose $N = 150$ and $M = 100$. Then we have to check at least all choices of 100 out of 150 coordinates as the 0 coordinates, and the number of choices is

$$\binom{150}{100} \geq 10^{40},$$

a hopelessly large number, even for the fastest computer.

The idea behind the simplex algorithm is illustrated in Figure 2.e, which shows the value of the objective function $g(x, y)$ at only 3 extreme points. Since the objective function $g(x, y)$ is an affine function, the level set $g(x, y) = 6$ is a line in the plane. The line cuts the plane into two parts, and the location of the extreme points where $g(x, y) = 3$ shows that $g(v) \leq 6$ at all extreme points v of the octagon. Thus the problem has been solved by finding a extreme point v and a set of neighboring extreme points $w \in W(v)$ such that $g(v) > g(w)$.

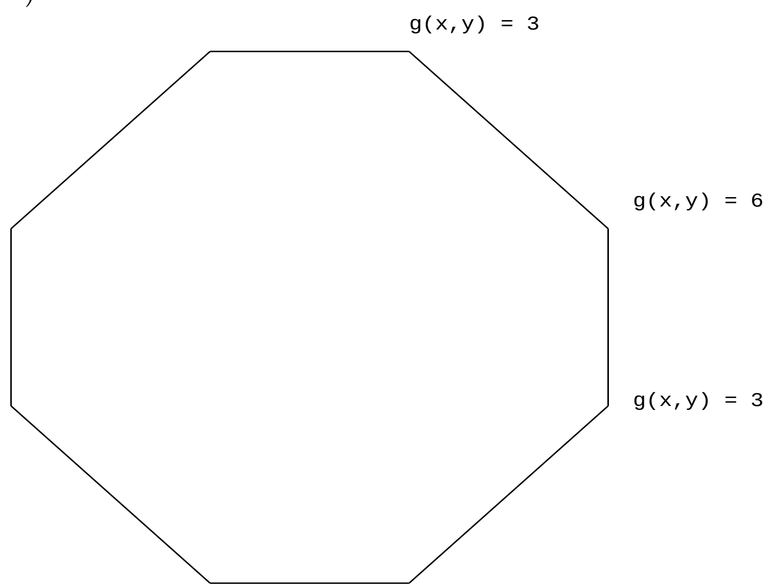

Figure 2.e: Simplex example

To extend this idea to a general linear program, we need the right notion for the set $W(v)$ of neighboring extreme points, and we then need to prove that the information $g(v) > g(w)$ for $w \in W$ is sufficient to solve the problem.

Given these results, the simplex algorithm simply says to start at some extreme point, then continue moving to neighboring extreme points so that the objective function always increases. Stop when no further increase is possible.

To simplify our analysis a bit we will make two assumptions about the constraint set F. First we assume that F is closed and bounded. This is a common situation, as is discussed in the exercises.

The second assumption concerns the extreme points of the constraint set F. The constraints are given by the matrix equation

$$AX = B,$$

where A is an $M \times N$ matrix with $M < N$ independent rows. Let $K = N - M$, and for each set S of K distinct indices $n(1), \dots, n(K)$ consider the augmented system

$$\begin{aligned} AX &= B \\ X_{n(1)} &= 0 \\ &\vdots \\ X_{n(K)} &= 0, \end{aligned}$$

which we abbreviate as

$$A_S X = \begin{pmatrix} B \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The assumption is

H: each extreme point X_0 has exactly $K = N - M$ coordinates equal to 0, the rest being positive.

This assumption is not always true, but it is typical (for each $M < N$ it holds for an open dense set of $M \times N$ matrices). As an example, consider the single constraint in $\mathbb{R}^3$,

$$x_1 + x_2 + x_3 = 1.$$

The extreme points are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, which have precisely 2 coordinates equal to 0.

Suppose that assumption H holds and v is an extreme point of F. We then define the set $W(v)$ of extreme points neighboring v to be the set of extreme points w of F such that the set Z_v of 0 coordinates for v and the set Z_w of 0 coordinates for w have exactly $K-1$ coordinates in common.

Consider the example $x_1+x_2+x_3=1$ above, where the extreme points are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, If $v=(1,0,0)$, then the neighboring extreme points are $w_1=(0,1,0)$ and $w_2=(0,0,1)$, since both w_1 and w_2 agree with v in the position of $K-1=1$ value 0.

Let's consider the existence of neighboring extreme points.

Theorem 2.14. *Suppose F is closed and bounded, and condition H holds. Then every extreme point v has exactly $K=N-M$ distinct neighboring extreme points w_k. The vectors w_k are linearly independent in $\mathbb{R}^N$*

The justification for the simplex algorithm is based on the following result.

Theorem 2.15. *Suppose F is closed and bounded and assumption H holds. Assume that v is an extreme point of F such that $g(v)>g(w)$ for all $w\in W(v)$. Then for every extreme point $u\in F$ we have $g(v)>g(u)$ if $u\neq v$.*

In overview then, here is a (simplified) version of the simplex algorithm.

2.5.1 Simplified Simplex Algorithm

1. Put the linear programming problem into equality linear program (ELP) form by introducing slack variables if needed. The resulting problem has the form maximize $g(x_1,\dots,x_N)=\sum c_n x_n$ for $X=(x_1,\dots,x_N)$ in the feasible set F defined by the constraints

$$x_n \geq 0$$

and

$$AX=B, \quad X\in\mathbb{R}^N, \quad B\in\mathbb{R}^M.$$

That is

$$X=\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \quad B=\begin{pmatrix} b_1 \\ \vdots \\ b_M \end{pmatrix},$$

and where $A=(a_{mn})$ is an $M\times N$ matrix with $M<N$ linearly independent rows.

2. Pick an extreme point $v \in F$ by picking $K = N - M$ coordinates $x_{n(1)}, \ldots, x_{n(K)}$ and finding a solution of

$$AX = B,$$

$$x_{n(1)} = 0, \ldots, x_{n(K)} = 0,$$

with coordinates $x_j > 0$ if j is not one of the $n(k)$.

3. Find the neighboring extreme points $w \in W(v)$ to v using the method of step 2, with the requirement that the coordinates which are 0 for w include exactly $K - 1$ of the coordinates which are 0 for v.

4. Compare the values of $g(v)$ and $g(w)$ for $w \in W(v)$. If $g(v)$ is the largest, stop; you have a solution. If not, replace v with the w having the largest value and return to step 3.

2.5.2 Qualifications

For most linear programming problems the simplified algorithm described above is effective, but there are several difficulties that must be dealt with by a polished algorithm. Here is a list of the problems.

(i) If the feasible set is unbounded, there may be no solution.

(ii) When searching for extreme points in steps 2 and 3, the system of equations may not have a unique solution (coefficient matrix is not invertible).

(iii) In step 4 you may find that $g(v)$ is the same as the largest $g(w)$.

2.6 Exercises

1. Show that the function $g = 2x + 3y$ has no maximum when subjected to the constraints $x - y = 3$, $x \geq 0$, $y \geq 0$, and conclude that linear programs are not always solvable.

2. Suppose $U_1 = (0,0)$, $U_2 = (1,0)$, and $U_3 = (0,1)$. Find the set of convex combinations

$$w_1U_1 + w_2U_2 + w_3U_3, \quad w_j \geq 0, \quad w_1 + w_2 + w_3 = 1.$$

3. Solve the following linear programming problems. Use the idea of Theorem 2.13 that the maximum must occur at an extreme point, and the extreme points are feasible solutions ($x_n \geq 0$) of the constraint equations with at least 2 components equal to 0.

(a) Maximize $x_1 + 2x_2 + 3x_3 + 2x_4$ subject to the constraints

$$x_1 + 2x_2 + x_3 + x_4 = 4,$$

$$2x_1 + x_2 + 3x_3 + x_4 = 3,$$

and all $x_k \geq 0$.

(b) Maximize $x_1 - x_2 + 3x_3 + 2x_4$ subject to the constraints

$$x_1 + x_2 + x_3 + x_4 = 4,$$

$$x_1 + x_2 - x_3 - x_4 = 3,$$

and all $x_k \geq 0$.

4. The text by Meerschaert introduces the following inequality program: maximize $P = 400x_1 + 200x_2 + 250x_3$, subject to

$$3x_1 + x_2 + 1.5x_3 \leq 1000,$$

$$0.8x_1 + 0.2x_2 + 0.3x_3 \leq 300,$$

$$x_1 + x_2 + x_3 \leq 625,$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

(a) By introducing slack variables x_4, x_5, x_6, rewrite this problem as an equality linear program.

(b) Show that the feasible set F for the equality program is bounded. (It is also closed.)

(c) Find one extreme point for this program. Using the idea of neighboring extreme points from the discussion of the simplex algorithm, find two neighboring extreme points. Compare the values of the profit at these 3 extreme points.

5. Suppose one of the constraints

$$\sum_{n=1}^{N} a_{mn} x_n = b_m,$$

in the ELP has all coefficients $a_{mn} > 0$. Show that the feasible set F is bounded. Can you find a weaker condition that will still guarantee that the feasible set is bounded.

6. Suppose we have an inequality linear program of the form maximize

$$g(x_1, \ldots, x_N) = \sum_{n=1}^{N} c_n x_n,$$

subject to constraints $x_n \geq 0$ for $n = 1, \ldots, N$ and

$$a_{11}x_1 + \cdots + a_{1,N}x_N \leq b_1,$$

$$\vdots,$$

$$a_{M1}x_1 + \cdots + a_{M,N}x_N \leq b_M.$$

Show that the feasible set for the corresponding equality linear program is bounded if $a_{m,n} > 0$ for $m = 1, \ldots, M$ and $n = 1, \ldots, N$. Formulate a weaker condition that will still guarantee that the feasible set is bounded.