Final Examination   Math 448/548
Professor Carlson

Show all your work. All questions have equal value.

1. a) Show that the perimeter of a rectangular sheet of fixed area is minimized if the sheet is square.

b) Find the values of x and y which minimize the function

$$f(x,y) = x^2 + y^2 - 2x - 4y + 16.$$

Justify the claim that the point you found is a minimum.

c) Minimize the function of part b) if x and y are required to satisfy $x^2 + y^2 = 1$.

2. An automobile manufacturer makes a profit of $1,500 on the sale of a certain model. It is estimated that for every $100 of rebate, sales increase by 15%.

a) What amount of rebate will maximize profit?

b) Compute the sensitivity of your answer to the 15% assumption. Consider both the amount of rebate and the resulting profit.

3. Consider the following linear programming problem.

Maximize $x_1 + 2x_2 + 3x_3 + 2x_4$ subject to the constraints

$$x_1 + 2x_2 + x_3 + x_4 = 4,$$

$$2x_1 + x_2 + 3x_3 + x_4 = 3,$$

and all $x_k \ge 0$.

a) Find an EXPLICIT bound on the coordinates of the feasible points. Show your reasoning.

b) Solve the problem of part (a) by using the idea that the maximum must occur at an extreme point, and the extreme points are feasible solutions ($x_n \ge 0$) of the constraint equations with at least 2 components equal to 0.

c) What does it mean to say that $\Omega \subset R^N$ is convex?

4. Consider a dynamical system of the form

$$\frac{dx}{dt} = f(x,y),$$

$$\frac{dy}{dt} = g(x, y).$$

a) Define equilibrium point. How do you find them?

b) Suppose (x_0, y_0) is an equilibrium point. What matrix would you use to approximate the dynamical system near the equilibrium point? What do the eigenvalues of this matrix tell you about the stability of the equilibrium point?

c) Use the eigenvalue method to determine the stability of the equilibrium point $(0, 0)$ for the linear system

$$\frac{dx}{dt} = 4x + y,$$

$$\frac{dy}{dt} = x - 8y.$$

5. Random arrivals during a fixed time interval T are often modeled with a Poisson distribution,

$$p(X = n) = e^{-\lambda T} \frac{(\lambda T)^n}{n!}.$$

We will suppose that bus arrivals at a bus stop are being modeled, and that T is one hour.

a) You count bus arrivals between 7 am and 8 am every work day for 4 weeks (20 days); the total is 100 buses. What is the appropriate estimate of the expected value μ of X. ?

b) Show that the expected number of arrivals per hour is

$$\mu = E(X) = \lambda T.$$

c) Suppose that X_n is the number of bus arrivals between 7 am and 8 am on day n. The central limit theorem says that

$$Prob\{\frac{X_1 + \ldots + X_n - n\mu}{\sigma\sqrt{n}} \leq t\} \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx.$$

Assume that the variance of X_n is also λT. Express (as an integral) the central limit theorem estimate of the probability that in the next 20 work days the total number of bus arrivals will be between 90 and 110. (Do not try to evaluate the integral.)