Chapter 4 # 1,2,5,7

1. (Example 4.1) The competition between two types of trees is modeled by the dynamical system

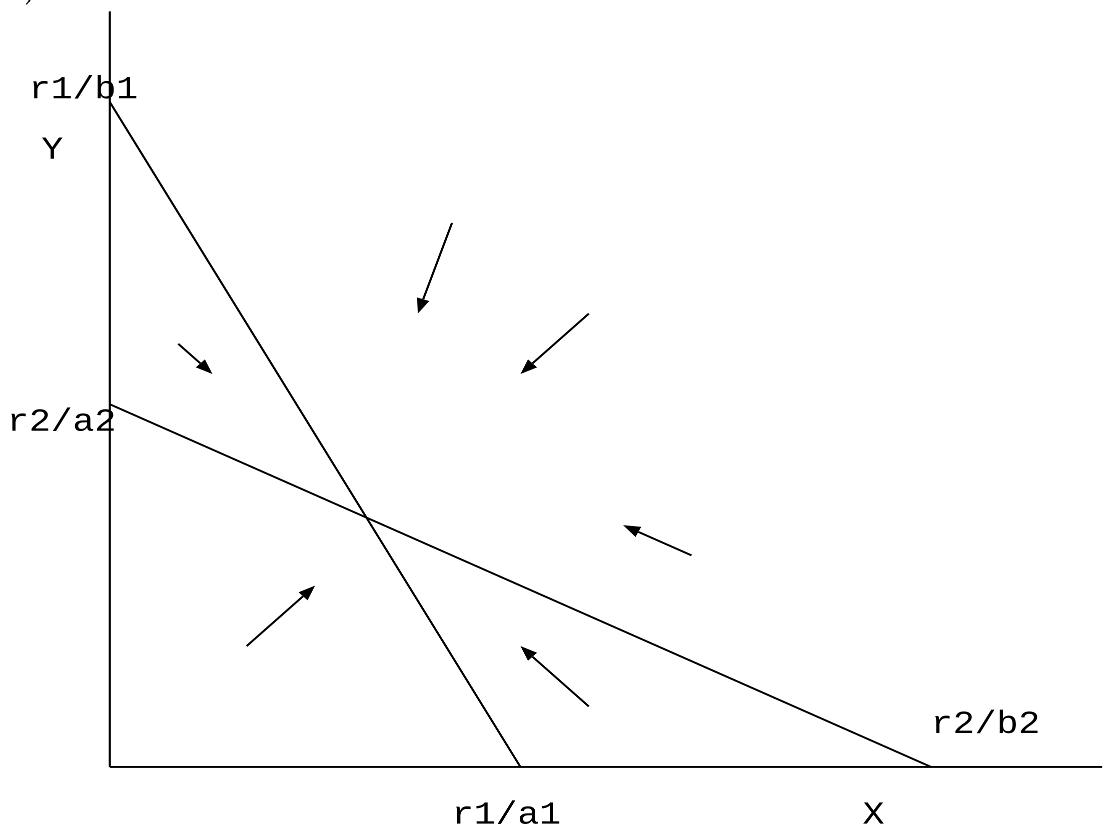
$$\frac{dx}{dt} = r_1 x - a_1 x^2 - b_1 xy,$$

$$\frac{dy}{dt} = r_2 y - a_2 y^2 - b_2 xy.$$

Assume that

$$\frac{r_2}{a_2} < \frac{r_1}{b_1}, \quad \frac{r_1}{a_1} < \frac{r_2}{b_2}.$$

a) Draw the vector field for this model.



b) Classify the four equilibrium points as stable or unstable.

Based on the figure, the equilibrium at the intersection of the two lines is stable, the others are unstable.

c) Can the two species exist in stable equilibrum?

Yes.

d) What does the model predict if hardwoods are reduced to a small population by logging?

All solutions that start with $x > 0$ and $y > 0$ approach the stable equilibrium.

2. Reconsider problem 1, but now assume that

$$\frac{r_2}{a_2} < \frac{r_1}{b_1}, \quad \frac{r_1}{a_1} \geq \frac{r_2}{b_2}.$$

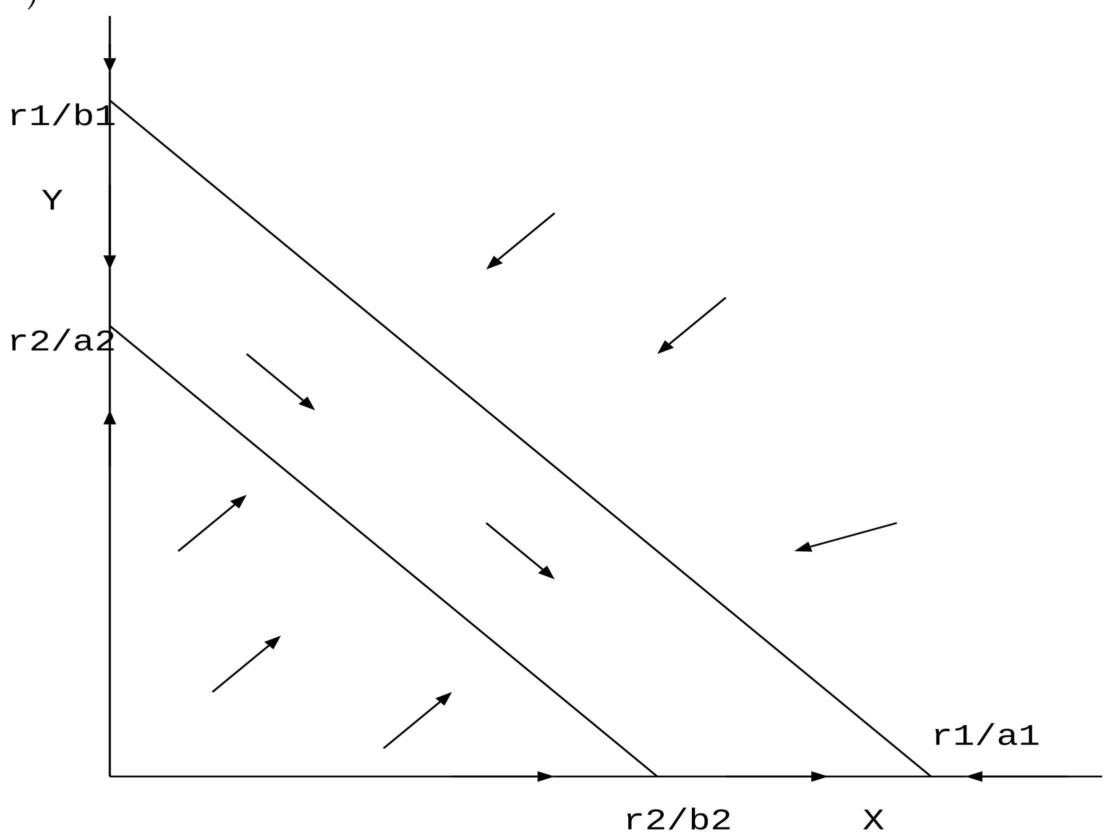
a) Locate the equilibrium points satisfying $x \geq 0$, $y \geq 0$.

Since the lines don't intersect, the only equilibrium points are on the coordinate axes:

$$x = 0, y = r_2/a_2, \quad y = 0, x = r_1/a_1.$$

There is also an equilibrium point at $x = 0$, $y = 0$.

b) Draw the vector field.



c) Classify the equilibrium points as stable or unstable.

The point $y = 0, x = r_1/a_1$ is stable, the others are unstable.

d) What does the model predict if the populations start at equal levels?

Eventual extinction for the softwoods.

5. Replace the logistic model

$$g(P) = rP(1 - P/K)$$

of example 4.2 with

$$g(P) = rP\frac{P-c}{P+c}(1 - P/K).$$

The model in this case is

$$\frac{dx}{dt} = r_1 x \frac{x - c_1}{x + c_1}(1 - x/K_1) - \alpha xy,$$

$$\frac{dy}{dt} = r_2 y \frac{y - c_2}{y + c_2}(1 - y/K_2) - \alpha xy.$$

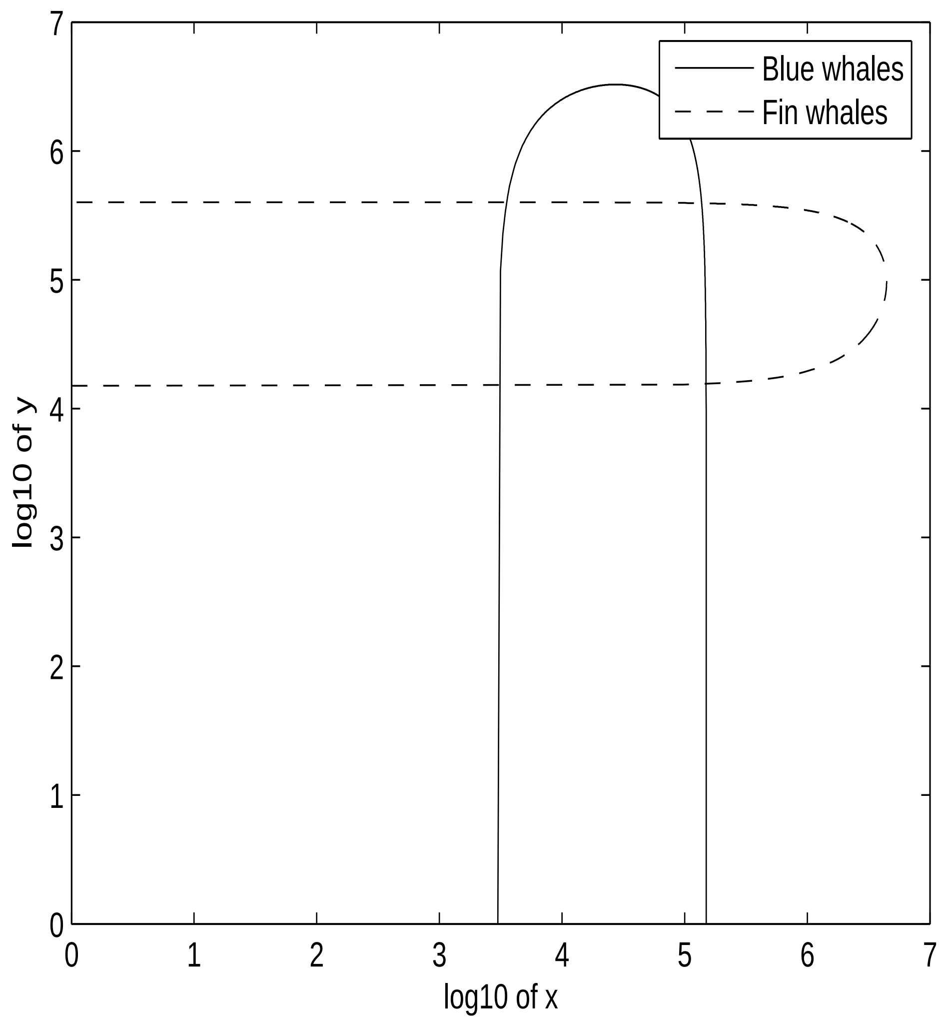
The parameter values are

$$\alpha = 10^{-8}, \quad c_1 = 3,000, \quad c_2 = 15,000,$$

$$r_1 = .05, \quad r_2 = .08, \quad K_1 = 150,000, \quad K_2 = 400,000.$$

a) Can the two species coexist?

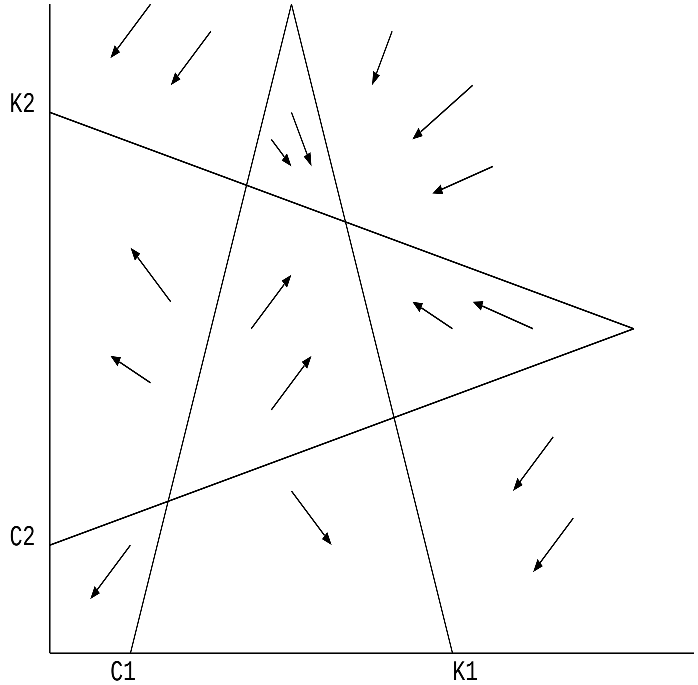
Looking for equilibria, we set the two derivatives equal to 0, divide in the first equation by x, in the second by y, and plot the results. Positive equilibria are indicated by the intersections of the curves. To help with scaling, the plots are of the log base 10 of the x and y values when both are positive.



There are equilibria with both populations positive. The equilibria and stability are discussed next.

b) Sketch the vector field.

For ease of drawing arrows we use the following schematic figure.



c) What does the model predict if the populations start at 5,000 blue whales and 70,000 fin whales?

Both populations are in their respective growth regions, so the populations grow toward the stable equilibrium at the upper right of the figure.

d) What happens if the minimum viable population of blue whales is 10,000? The blue whales simply decrease to 0.

7. Blue whales feed on krill. The maximum sustainable population of krill is 500 tons per acre. Krill populations grow at 25 percent per year. The presence of 500 tons per acre of krill increases the blue whale population growth rate by 2 percent per year. The presence of 150,000 blue whales decreases the krill growth rate by 10 percent per year.

Although one might try to justify a model along the lines of problem 1, I read the problem to give the following model.

$$\frac{dx}{dt} = (r_1 - .1\frac{y}{150,000})x(1 - x/K_1),$$

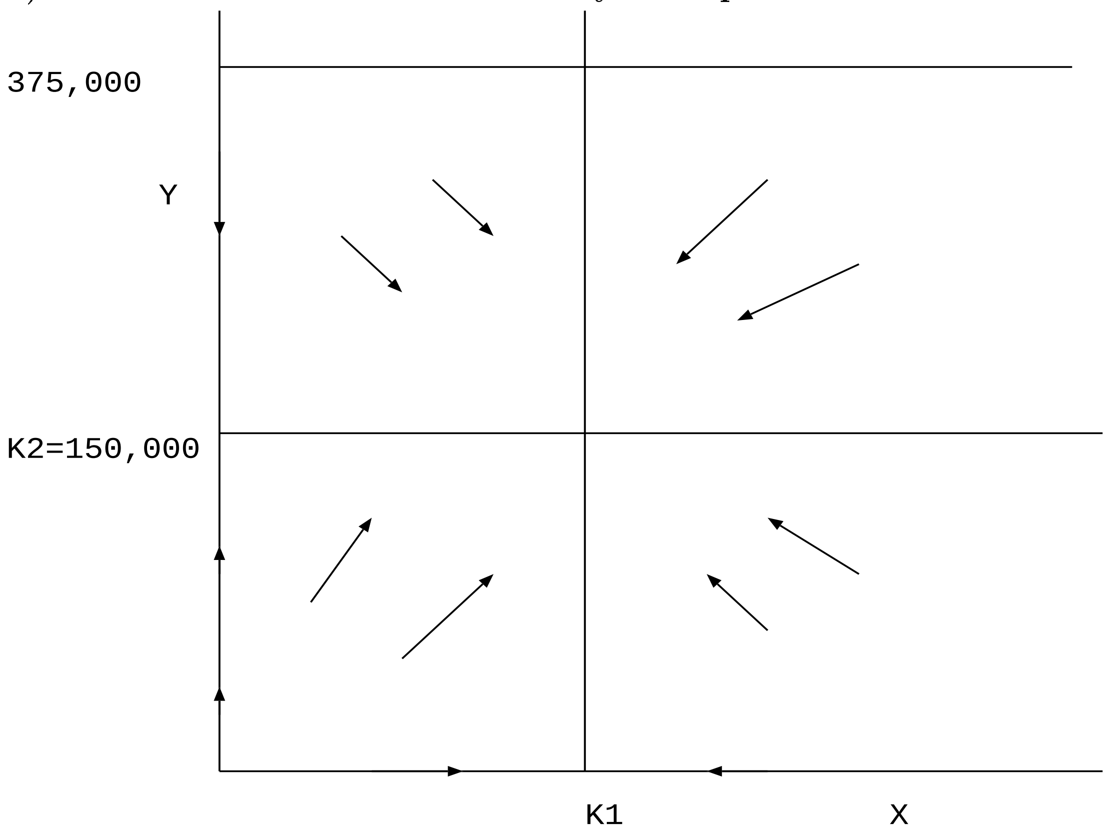
$$\frac{dy}{dt} = (r_2 + .02\frac{x}{500})y(1 - y/K_2).$$

here x is the krill population in tons per acre, while y is the whale population.

a) Can the blue whales and krill exist in equilibrium?

Yes, there is a stable equilibria with both populations positive.

b) Draw the vector field and classify the equilibria.



The main equilibrium point occurs when both populations are at their carrying capacities, $x = K_1$ and $y = K_2$. Only this central equilibrium is stable.

c) Describe the evolution if we start with 5,000 blue whales and 750 tons per acre of krill.

The krill population is reduced, while the whales grow. Eventually the two populations approach the stable equilibrium.

d) Study the sensitivity of part c) to the 25 percent growth rate.