3.7 Linear programming solutions

1. *Show that the function $g = x + y$ has no maximum when subjected to the constraints $x - y = 5$, $x \ge 0$, $y \ge 0$, and conclude that linear programs are not always solvable.*

Write the constraint $x - y = 5$ as $y = x - 5$. Along this line we can write $g(x, y) = G(x) = x + (x - 5) = 2x - 5$. For $x \ge 0$ the function $2x - 5$ has no maximum, so the linear programming problem to maximize $x + y$ subject to the constraint $x - y = 5$ has no solution.

2. *Suppose $U_1 = (0, 0)$, $U_2 = (1, 0)$, and $U_3 = (0, 1)$. Find the set K of convex combinations*

$$w_1U_1 + w_2U_2 + w_3U_3, \quad w_j \ge 0, \quad w_1 + w_2 + w_3 = 1.$$

Let F be the set (a triangular region) in the $x - y$ plane given by

$$F = \{(x, y) \mid x \ge 0, y \ge 0, x + y \le 1\}.$$

We'll show that F and K are the same.

First, any point in K can be written as

$$w_1U_1 + w_2U_2 + w_3U_3 = (w_2, w_3), \quad w_j \ge 0.$$

Since in addition $w_1 + w_2 + w_3 = 1$, we have $w_2 + w_3 \le 1$. Thus $K \subset F$.

Next, suppose $(w_2, w_3) \in F$. Define $w_1 = 1 - w_2 - w_3$. Then $w_1 \ge 0$, $w_1 + w_2 + w_3 = 1$, and

$$(w_2, w_3) = w_1U_1 + w_2U_2 + w_3U_3.$$

Thus $(w_2, w_3) \in K$, and the sets K and F are equal.

3. *Solve the following linear programming problem. Use the idea of Theorem 3.4.5 that the maximum must occur at an extreme point, and the extreme points are feasible solutions ($x_n \ge 0$) of the constraint equations with at least 2 components equal to 0.*

Maximize $g(x_1, \dots, x_4) = x_1 - x_2 + 3x_3 + 2x_4$ subject to the constraints

$$x_1 + x_2 + x_3 + x_4 = 4,$$

$$x_1 + x_2 - x_3 - x_4 = 3,$$

and all $x_k \ge 0$.

First, the problem has a solution since the first constraint forces the feasible set to be bounded.

We check for solutions with at least two components equal to 0. Here are the solutions.

$$x_1 = 0, x_2 = 0, \quad \textit{no solutions},$$

$$x_1 = 0, x_3 = 0, \quad x_2 = 7/2, x_4 = 1/2,$$

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$$x_2 = 0, x_4 = 0, \quad x_1 = 7/2, x_3 = 1/2,$$

$$x_3 = 0, x_4 = 0, \quad \textit{no solutions}.$$

The four cases with solutions are all feasible, so these are extreme points. Checking these cases we find

$$x_1 = 0, x_3 = 0, x_2 = 7/2, x_4 = 1/2, \quad g = -7/2 + 1 = -5/2,$$

$$x_1 = 0, x_4 = 0, x_2 = 7/2, x_3 = 1/2, \quad g = -7/2 + 3/2 = -2,$$

$$x_2 = 0, x_3 = 0, x_1 = 7/2, x_4 = 1/2, \quad g = 7/2 + 1 = 9/2,$$

$$x_2 = 0, x_4 = 0, x_1 = 7/2, x_3 = 1/2, \quad g = 7/2 + 3/2 = 5.$$

The maximum value is 5, which occurs at $x_2 = 0, x_4 = 0, x_1 = 7/2, x_3 = 1/2$.

4. *The text by Meerschaert introduces the following inequality program: maximize* $P = 400x_1 + 200x_2 + 250x_3$, *subject to*

$$3x_1 + x_2 + 1.5x_3 \leq 1000,$$

$$0.8x_1 + 0.2x_2 + 0.3x_3 \leq 300,$$

$$x_1 + x_2 + x_3 \leq 625,$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

(a) By introducing slack variables x_4, x_5, x_6, *rewrite this problem as an equality linear program.*

$$3x_1 + x_2 + 1.5x_3 + x_4 = 1000,$$

$$0.8x_1 + 0.2x_2 + 0.3x_3 + x_5 = 300,$$

$$x_1 + x_2 + x_3 + x_6 = 625,$$

$$x_1 \geq 0, \ldots, x_6 \geq 0.$$

(b) *Show that the feasible set F for the equality program is bounded. Why is it also closed?*

Since all of the variables are nonnegative, the first constraint gives

$$x_1 \leq 1000/3, x_2 \leq 1000, x_3 \leq 1000/1.5, x_4 \leq 1000,$$

and the remaining constraints give

$$x_5 \leq 300, x_6 \leq 625.$$

Thus F is bounded.

Showing that F is closed requires some analysis. The feasible set F is the intersection of the sector S with all coordinates nonnegative and the sets K_m where the constraint functions $\sum_n a_{mn} = C_m$ are constant. The intersection of closed sets is closed, so we only have to show that each of the sets is closed. The constraint functions $\sum_n a_{mn}$ are continuous, and the set K_m where a continuous function is constant is a closed set. Suppose $X(k) = (x_1(k), \ldots, x_N(k))$ is a sequence in $\mathbb{R}^N$ which converges to $Y = (y_1, \ldots, y_N)$. If each component of $X(k)$ is nonnegative, then each component of Y is nonnegative,

$$y_n = \lim_{k \to \infty} x_n(k) \geq 0.$$

Thus the sector S is closed.

(c) *Find one extreme point for this program. Using the idea of neighboring extreme points from the discussion of the simplex algorithm, find two neighboring extreme points. Compare the values of the profit at these 3 extreme points.*

Try $x_1 = x_2 = x_3 = 0$. This gives the feasible values

$$x_4 = 1000, x_5 = 300, x_6 = 625, \quad P = 0,$$

To find neighboring extreme points we try $x_2 = x_3 = x_4 = 0$ to get

$$x_1 = 1000/3, x_5 = 100/3, x_6 = 875/3, \quad P = \frac{400,000}{3},$$

which has the biggest profit of these three points, and $x_1 = x_3 = x_6 = 0$ to get

$$x_2 = 625, x_4 = 375, x_5 = 125, \quad P = 125,000.$$

6. *Suppose we have an inequality linear program of the form maximize*

$$g(x_1, \ldots, x_N) = \sum_{n=1}^{N} c_n x_n,$$

subject to constraints $x_n \geq 0$ *for* $n = 1, \ldots, N$ *and*

$$a_{11}x_1 + \cdots + a_{1,N}x_N \leq b_1,$$

$$\vdots,$$

$$a_{M1}x_1 + \cdots + a_{M,N}x_N \leq b_M.$$

Show that the feasible set for the corresponding equality linear program is bounded if $a_{m,n} > 0$ *for* $m = 1, \ldots, M$ *and* $n = 1, \ldots, N$. *Formulate a weaker condition that will still guarantee that the feasible set is bounded.*

The corresponding equality linear program has the form

$$a_{11}x_1 + \cdots + a_{1,N}x_N + x_{N+1} = b_1,$$

$$\vdots,$$

$$a_{M1}x_1 + \cdots + a_{M,N}x_N + x_{N+M} = b_M.$$

with $x_n \geq 0$ for $n = 1, \ldots, N, \ldots, N+M$. Because all $x_n \geq 0$, and $a_{m,n} > 0$ the first equation gives

$$x_n \leq b_1/a_{1n}, \quad n = 1, \ldots, N.$$

The equations also yield

$$x_{N+m} \leq b_m, \quad m = 1, \ldots, M.$$

Thus all coordinates are bounded.

For the weaker condition, notice that we did not need to have all $a_{mn} > 0$. It is sufficient to have a single value of k with $a_{kn} > 0$ for $n = 1, \ldots, N$. As in the above argument, this would first imply that

$$x_n \leq b_1/a_{kn}, \quad n = 1, \ldots, N.$$

Now solving for x_{N+m} in the m-th equation, we can get a bound on x_{N+m} for each m. Thus all coordinates will be bounded.