

Math 448/548 - Mathematical Modeling Professor Carlson
Homework 1 Solutions

1. An automobile manufacturer makes a profit of \$1500 on the sale of a certain model. It is estimated that for every \$100 in rebate, sales increase by 15% .

a) What amount of rebate will maximize profit?

$$\text{Rebate} = 100x$$

$$\text{Profit per car} = 1500 - 100x$$

$$\text{Number of cars sold} = N_0(1 + .15x)$$

$$P = \text{Profit} = (\text{Number of cars sold}) * (\text{Profit per car})$$

$$= N_0(1 + .15x)(1500 - 100x).$$

To maximize profit, look for critical points where $P' = 0$.

$$P' = N_0[.15(1500 - 100x) + (1 + .15x)(-100)] = 0.$$

The only critical point is $12.5/3$; since $P(x)$ is a quadratic polynomial with coefficient of x^2 negative, this is a maximum.

The optimum rebate is $100x = 1250/3 \simeq 417$ dollars.

b) Compute the sensitivity of your answer to the 15% assumption. Consider both the amount of rebate and the profit.

Sensitivities are calculated as follows. For profit

$$P(x, c) = N_0(1 + cx)(1500 - 100x).$$

$$S(P, c) = \frac{dP}{dc} \frac{c}{P} = x(1500 - 100x) \frac{c}{P} = \frac{125}{30} \frac{0.15}{(1 + 0.15 * 125/30)} = 0.385$$

For rebate we find x as a function of c .

$$\frac{dP(x, c)}{dx} = 0 = N_0[c(1500 - 100x) + (1 + cx)(-100)].$$

This gives

$$X(c) = \frac{15c - 1}{2c}.$$

The sensitivity is

$$S(X, c) = \frac{dX}{dc} \frac{c}{X} = \frac{30c - 2(15c - 1)}{4c^2} \frac{c}{X} = \frac{1}{2cX} = \frac{1}{.3 * 4.17} \simeq \frac{1}{1.2} = .8.$$

c) Suppose the rebates only generate a 10% increase in sales per \$ 100. What is the effect? What if the response is somewhere between 10 and 15 percent?

If $c = .1$ instead of $.15$ this is a 33% decrease in the value of c . The profit changes approximately by

$$S(P, c) * .33 = .385 * .33 \simeq .13,$$

That is profit drops by about 13 percent.

A similar sensitivity analysis works to predict rebate should be reduced by about $.8 * .33 \simeq .26$, or 26 percent. The new rebate should be about \$ 309.

d) Under what circumstances would a rebate offer cause a reduction in profit?

I interpret the question to mean is there a value of c for which even a small rebate is not profitable. This will happen if there is a value of c such that

$$(1500 - 100x)(1 + cx) < 1500, \quad x > 0.$$

This happens if $c \leq 1/15 \simeq .067$.

5. It is estimated that the growth rate of the fin whale population (per year) is $rx(1 - x/K)$, where $r = 0.08$ is the intrinsic growth rate, $K = 400,000$ is the maximum sustainable population, and x is the current population, now around 70,000. It is further estimated that the number of whales harvested per year is about $10^{-5}Ex$, where E is the level of fishing effort in boat days. Given a fixed level of effort, population will eventually stabilize at the level where growth rate equals harvest rate.

a) What level of effort will maximize the sustained harvest rate?

First of all we equate growth rate and harvest rate to determine the stable population:

$$rx(1 - x/K) = 10^{-5}Ex,$$

or

$$x = K(1 - 10^{-5}E/r).$$

We want to maximize $10^{-5}Ex$, or more conveniently

$$Ex = EK(1 - 10^{-5}E/r)$$

as a function of E . Plugging in the values of K and r gives a max of $E = 4,000$.

b) Examine the sensitivity to the intrinsic growth rate. Consider both optimum level of effort and resulting population.

The population is

$$x = K(1 - 10^{-5}E/r).$$

With the above values for K , r and E we find $x = 2 * 10^5$. The associated sensitivity is

$$S(x, r) = \frac{dx}{dr} \frac{r}{x} = 1.$$

The optimum level of effort as a function of r is

$$E_{max} = 5 * 10^4 r.$$

The sensitivity is

$$\frac{dE_{max}}{dr} \frac{r}{E_{max}} = 5 * 10^4 * .08/4000 = 1.$$

c) Examine the sensitivity to the maximum sustainable population. Consider both the optimum level of effort and the resulting population.

Again the population is

$$x = K(1 - 10^{-5}E/r) = K(1 - 10^{-2}4/.08) = K/2.$$

Thus

$$S(x, K) = \frac{dx}{dK} \frac{K}{x} = 1.$$

Also with the value $r = .08$ the optimum level of effort is 4,000 independent of K , so the sensitivity here is 0.