1.5 Elementary Matrices

1.5.2 *Find a row operation that will convert the given elementary matrix to an identity matrix.*

$$(a) \quad \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \quad (b) \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

(a) Add 3 times the first row to the second row. (b) Multiply the third row by 1/3.

1.5.4 Let

$$B = \begin{pmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{pmatrix}.$$

Is it possible to find an elementary matrix E such that $EB = C$? No. Let's consider the possibilities. The elementary row operations are (i) multiply a row by a nonzero constant, (ii) interchange two rows, (iii) add c times row i to row j. Since the first row of B does not appear in C, we cannot use operation (ii) to go from B to C. On the other hand, operations (i) and (iii) change only one row of B. Since B and C have two rows that are not equal, we can't convert B to C with an operation that changes only one row.

1.5.5 Consider the matrices

$$(a) \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (b) \quad \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, \quad (c) \quad \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

If a 2×2 matrix is multiplied on the left by one of these matrices, what elementary row operation is performed on that matrix?

(a) Exchange the two rows.

(b) No elementary operation. Multiplication by (b) changes two rows by different nonzero multiples.

(c) Add -2 times row 1 to row 2.

1.5.7 Find the inverse of

$$(a) \quad A = \begin{pmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{pmatrix}.$$

$$\begin{pmatrix} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 4 & -10 & 1 & -3 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & 5 & -7 & -4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 7/10 & 4/10 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 3/2 & -11/10 & -6/5 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{pmatrix}.$$

Thus

$$A^{-1} = \begin{pmatrix} 3/2 & -11/10 & -6/5 \\ -1 & 1 & 1 \\ -1/2 & 7/10 & 2/5 \end{pmatrix}.$$

$$(c) \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}.$$

1.5.9 Find the inverse of

$$(a) \quad A = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1/k_1 & 0 & 0 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 0 & 0 & 1/k_4 \end{pmatrix}.$$

$$(b) \quad A = \begin{pmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1/k_4 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 1/k_1 & 0 & 0 & 0 \end{pmatrix}.$$

1.6 Further results on invertibility

1.6.2 *Solve the system*

$$4x_1 - 3x_2 = -3,$$

$$2x_1 - 5x_2 = 9.$$

Write in matrix form $Ax = b$, with

$$A = \begin{pmatrix} 4 & -3 \\ 2 & -5 \end{pmatrix}.$$

Then

$$A^{-1} = \frac{-1}{14} \begin{pmatrix} -5 & 3 \\ -2 & 4 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{-1}{14} \begin{pmatrix} -5 & 3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} -3 \\ 9 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}.$$

1.6.23 *Let $Ax = 0$ be a homogeneous system of n equations in n unknowns with only the trivial solution. Show that if k is any positive integer, then $A^k x = 0$ also has only trivial solutions.* Since $Ax = 0$ has only the trivial solution $x = 0$, the matrix A is invertible. If A is invertible then A^k is invertible, with inverse $(A^{-1})^k$. Since A^k is invertible, the system $A^k x = 0$ also has only the trivial solution.