Solutions for Test 2   Math 313   Professor Carlson

1. (a) If $v = (1,2,3)$, find $\|v\|$. *Find a vector U of length 1 in the same direction as v.* Multiply v by 1/norm:

$$\|v\|^2 = 1 + 4 + 3 = 14, \quad U = \frac{1}{\sqrt{14}}(1,2,3).$$

(b) *Find a nonzero vector w_1 that is orthogonal to $w_2 = (1,0,1)$. Show that they are orthogonal.* Orthogonal means $w_1 \bullet w_2 = 0$. An easy choice is $w_1 = (0,1,0)$, since $w_1 \bullet w_2 = 0 + 0 + 0 = 0$.

(c) *Suppose $v = (1,1,1)$ and $w = (1,0,-1)$. Find $\cos(\theta)$, where θ is the angle between these vectors.*

$$\cos(\theta) = \frac{v \bullet w}{\|v\|\|w\|}.$$

In this case $v \bullet w = 0$, so $\cos(\theta) = 0$.

2. *Suppose $u = (1,1,0)$ and $v = (0,1,-1)$.*
(a) Find $w = u \times v$.

$$w = u \times v = \det \begin{pmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= i(-1) - j(-1) + k = (-1,1,1).$$

(b) *Find $\sin(\theta)$, where θ is the angle between u and v.* For $0 \le \theta < \pi$

$$\|u \times v\| + \|u\|\|v\| \sin(\theta),$$

so

$$\sin(\theta) = \frac{\sqrt{3}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{3}}{2}.$$

(c) *Find $\cos(\phi)$, where ϕ is the angle between u and w.* Since u and w must be orthogonal, $\cos(\phi) = 0$.

(d) *Suppose $U, V, W \in R^3$ and*

$$U \times W = V \times W.$$

Can we conclude that $U = V$*? Explain.* No. If $U = (2, 0, 0)$, $V = (3, 0, 0)$, and $W = (1, 0, 0)$, then

$$U \times W = V \times W = (0, 0, 0).$$

3. (a) *The Cauchy - Schwarz inequality says that for vectors* u *and* v *in* R^n*,*

$$|u \bullet v| \leq \|u\|\|v\|.$$

Use the Cauchy-Schwarz inequality to prove that for all real values of a*,* b*, and* θ*,*

$$(a\cos(\theta) + b\sin(\theta))^2 \leq a^2 + b^2.$$

Take $u = (a, b)$ and $v = (\cos(\theta), \sin(\theta))$ to get

$$(a\cos(\theta) + b\sin(\theta))^2 = |u \bullet v|^2 \leq \|u\|^2\|v\|^2 = a^2 + b^2.$$

(b) *Show that if* u *and* v *are orthogonal vectors in* R^n*, then*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

If u and v are orthogonal vectors in R^n, then $u \bullet v = 0$. Thus

$$\|u + v\|^2 = (u + v) \bullet (u + v) = u \bullet u + 2u \bullet v + v \bullet v$$

$$= u \bullet u + v \bullet v = \|u\|^2 + \|v\|^2.$$

4. (a) *Find the standard matrix* A *for the linear transformation* $T : R^3 \to R^3$ *defined by*

$$w_1 = 2x_1 + 3x_2 + 4x_3,$$

$$w_2 = x_1 + x_2 + x_3,$$

$$w_3 = x_1 - 3x_2 + x_3.$$

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & -3 & 1 \end{pmatrix}.$$

(b) *Find the standard matrix for the linear transformation* $T : R^3 \to R^3$ *which takes a vector* $v = (x_1, y_1, z_1)$ *to its orthogonal projection onto the* $x - y$ *plane.*

The linear transformation $T : R^3 \to R^3$ is given by

$$w_1 = x_1,$$

$$w_2 = y_1,$$

$$w_3 = 0.$$

Thus

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. (a) *Suppose that* $T : R^n \to R^m$ *is a linear transformation from* R^n *to* R^m. *What does it mean to say that* T *is one-to-one? What does it mean to say that* T *is onto ?* T is one-to-one means that if $x \neq y$ then $T(x) \neq T(y)$, or equivalently the only time we have $T(x) = T(y)$ is when $x = y$. T is onto if for every $y \in R^m$ there is an $x \in R^n$ such that $T(x) = y$.

(b) *Suppose that* A *is an invertible* $n \times n$ *matrix, and the function* $T : R^n \to R^n$ *is given by* $T(x) = Ax$. *Show that* T *is one-to-one and onto. It is NOT sufficient to say that this is part of Theorem 4.3.4.*

Suppose $Ax = Ay$. Then $A(x - y) = 0$. Since A is invertible,

$$x - y = A^{-1}A(x - y) = A^{-1}0 = 0,$$

so $x = y$ and T is one-to-one.

Suppose $y \in R^n$. Take $x = A^{-1}y$. Then

$$Ax = AA^{-1}y = y,$$

and T is onto.