Solutions for Test 1 Math 313 Professor Carlson

SHOW ALL YOUR WORK

1. *(20 pts) Solve the following system of equations by Gaussian elimination.*

$$3x + y + z = 8,$$
$$x - y = -1,$$
$$x + y - z = 0.$$

It is convenient to put the last equation first. Then add a multiple of the first equation to get the form

$$x + y - z = 0.$$
$$-2y + z = -1,$$
$$-2y + 4z = 8,$$

and then get

$$x + y - z = 0.$$
$$-2y + z = -1,$$
$$3z = 9,$$

and so $z = 3$, $y = 2$, and $x = 1$.

2. *(20 pts) Suppose*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}.$$

(a) Compute AB.

$$AB = \begin{pmatrix} 3 & 11 \\ 0 & 10 \end{pmatrix}.$$

(b) Find B^{-1}.

$$B^{-1} = \frac{-1}{10} \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}.$$

(c) Find two 2×2 invertible matrices C and D such that $C + D$ is not invertible. How do you know C and D are invertible, and $C + D$ is not?

One example is

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C + D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

The determinants are $\det(C) = 1$, $\det(D) = -1$, $\det(C + D) = 0$. Since a matrix is invertible if and only if the determinant is not 0, we have the desired example.

3.*(20 pts) Suppose $k_1, \ldots, k_4$ are nonzero numbers. Find A^{-1}, and B^{-1} if*

$$A = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that

$$A^{-1} = \begin{pmatrix} 1/k_1 & 0 & 0 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 0 & 0 & 1/k_4 \end{pmatrix},$$

$$B^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1/k_4 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 1/k_1 & 0 & 0 & 0 \end{pmatrix}.$$

4.*(20 pts) Let*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find the cofactor matrix for A. Find $\det(A)$. Determine if A^{-1} exists. If it does, find it.

First

$$cof(A) = \begin{pmatrix} -1 & 2 & -1 \\ -2 & 1 & 1 \\ 4 & -2 & 1 \end{pmatrix}.$$

Then

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = -1 + 4 + 0 = 3.$$

Since $\det(A) \neq 0$, A^{-1} exists. It is

$$A^{-1} = \frac{1}{3}\begin{pmatrix} -1 & -2 & 4 \\ 2 & 1 & -2 \\ -1 & 1 & 1 \end{pmatrix}.$$

5. *(20 pts) a) Give three conditions that imply that an $n \times n$ matrix A is invertible.* Pick any three from Theorem 2.3.6.

b) Suppose A is an $n \times n$ matrix. Assume there are two $n \times 1$ matrices x and y with $x \neq y$, but $Ax = Ay$. Show that there is a nontrivial $n \times 1$ matrix z such that $Az = 0$. Is is possible for A to have an inverse?

Take $z = x - y$. Then $Az = Ax - Ay = 0$. It is not possible for A to have an inverse, by Theorem 2.3.6.

c) For which numbers x is the matrix

$$A = \begin{pmatrix} (x-1) & 1 & 1 & 2 \\ 0 & (x-2) & 3 & 1 \\ 0 & 0 & (x-3) & 0 \\ 0 & 0 & 0 & (x-4) \end{pmatrix}$$

singular? Why?

Since A is upper triangular,

$$\det(A) = (x-1)(x-2)(x-3)(x-4).$$

A is singular if and only if $\det(A) = 0$, which happens if and only if $x = 1, 2, 3, 4$.