The Mad Veterinarian on Mathematical Safari

SOLUTIONS AND COMMENTS ABOUT SOME OF THE PUZZLES

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HI all: I've written up solutions and discussion about Puzzle 1, and Puzzle 2 (through Question 2.5). I'd be very curious to hear from any of you who might use this sort of Problem Solving activity in your classroom about how it went over with the students.

Puzzles 3 and 4 (and the rest of Puzzle 2) follow the same ideas. You should be able to use Questions 3.1 through 3.3, and Questions 4.1, 4.2 if you want to more examples of trying to find "invariants". The additional questions in Puzzles 2,3,4 are somewhat more advanced, you might want to give them as food for thought to a student who's looking for more problems of this type. (If you or they have questions about how to start those additional questions, email me! abrams@math.uccs.edu .)

The website http://www.bumblebeagle.org/madvet/index.html is where the original puzzles are posted. The description of the puzzles given there is not the best. But the site author has written an applet to pictorially represent what each machine does. Your students might like to play around with that. But this site is definitely NOT necessary to visit in order to do these Mad Vet puzzles.

Puzzle 1: Answers and discussion

A Mad Veterinarian has created three animal transmogrifying machines.

Place a cat in the input bin of the first machine, press the button, and .... whirrr .... bing! Open the output bins to find four dogs and one mouse. The second machine can convert a dog into three cats and three mice, and the third machine can convert a mouse into a cat and a dog. Each machine can also operate in reverse (e.g., if you've got four dogs and one mouse, you can convert them into a cat.)

This Mad Vet has one cat.

Question 1.1:

Can he convert it into seven mice? (If so, show how. If not, explain why not. "Seven mice" means *exactly* seven mice, with no dogs or cats left over.)

Answer 1.1: YES. Here's one way:

- Use Machine 1 to convert the cat to 4 dogs and a mouse.
- Use Machine 2 to convert one of the dogs to 3 cats and 3 mice, so the result is 3 dogs, 3 cats, 4 mice.
- Use Machine 3 in reverse to convert 1 dog and 1 cat to 1 mouse, so the result is 2 dogs, 2 cats, 5 mice.
- Use Machine 3 in reverse again to convert 1 dog and 1 cat to 1 mouse, so the result is 1 dogs, 1 cat, 6 mice.
- Use Machine 3 in reverse again to convert 1 dog and 1 cat to 1 mouse, so the result is 7 mice.

Discussion / Questions: Is this the ONLY way to do it? Can it be done in fewer than five steps? [Definitely NOT the only way to do it, you can use some intermediate machines along the way. I think that five steps is minimum, but I don't have a proof. Maybe one of your students can argue why it's true?]

Question 1.2:

Can he convert it into a kennel of dogs, with no cats or mice left over? (If so, show how. If not, explain why not. F.Y.I, "kennel" is the word used for a group of dogs, while "pack" is the word used for a group of *wild* dogs …)

Answer 1.2: NO. Here's one way to see this. Consider the total number of cats and mice together that the Mad Vet might have in the clinic at any point in time. Notice that Machine 1 trades in a cat for a mouse (and some dogs, too), but if nothing else Machine 1 leaves the TOTAL number of cats and mice THE SAME. So does Machine 3. Machine 2, however, increases the total number of cats and mice together by 6. (Running it in reverse decreases this number by 6.) So the point is, regardless of what menagerie the Mad Vet might start with, the total number of cats and mice together can only change (increase or decrease) by a multiple of 6. So, for instance, if the Mad Vet starts with one cat, then the total number of cats and mice together is 1, so any other collection of animals which can be produced from one cat must have total number of cats and mice together to be: 1, 7, 13, 19, … etc. (In general, 1 + 6k for some whole number k.) So, in particular, the Mad Vet cannot change 1 cat into any number of just dogs, because "just dogs" would mean 0 cats and mice together, and 0 is not on the list of numbers 1, 7, 13, …

Discussion / Questions: There are other ways to argue this as well. For instance, the "parity" (evenness or oddness) of the total number of cats and mice together also does not change by any of the machines.

This idea of "something that stays the same even though lots of stuff is changing" is called an INVARIANT . So in this case, the total number of cats and mice together, up to a multiple of 6, is an invariant.

Puzzle 2: Answers and discussion

Another Mad Vet has these three machines:

Machine #1 turns one cat into one dog.
Machine #2 turns one dog into one cat, one dog, and one mouse
Machine #3 turns one mouse into one cat and one dog

This Mad Vet has one cat.

Question 2.1

Can she convert it into two cats? (If so, show how. If not, explain why not. Similar to the wording from Puzzle 1, and from now on, a phrase such as "two cats" means exactly two cats, with no dogs or mice left over.)

Answer 2.1: NO. Here's an invariant for this problem. Value each cat at $1, each dog at $1, and each mouse at $2. Then machines 1 and 3 do not change the total value of a collection of animals, regardless of what collection the Mad Vet might start with. Machine 2, on the other hand, increases the total value by $3 in the forward direction (decreases the total value by $3 in the reverse direction). So if the Mad Vet starts with any collection of animals, then regardless of what machines and in what order the Mad Vet runs animals through, the total value of the resulting collection has to be different from the value of the original collection by a multiple of $3. So for instance the original collection (one cat) is valued at $1. So the only new menageries the Mad Vet can cook up must be valued at $1, or $4, or $7, or … But a menagerie consisting of just 2 cats is valued at $2. So since $2 isn't on the allowed list, it can't be done.

Discussion / Questions: This is not obvious at first. Make sure your students wrestle with this for awhile before showing this 'value' approach. The dollar values to assign to the animals is somewhat arbitrary, but Machines 1 and 3 suggest that the value of cats and dogs be the same, and that the value of mice should be the sum of the value of cats and dogs.

This is the idea of "mod 3 arithmetic". To say that a number is on the list 1, 4, 7, 10, 13, … is the same as saying that the number is "congruent to 1 modulo 3".

Question 2.2

Can she convert it into three cats? (If so, show how. If not, explain why not.)

Answer 2.2: NO. Use the same invariant as in the previous problem. 3 cats would be worth $3, and $3 is not on the allowable list.

Question 2.3

Can she convert it into three cats and one dog? (If so, show how. If not, explain why not.)

Answer 2.3: YES.

- Use Machine 1 to convert the cat into a dog
- Use Machine 2 to convert the dog into a cat, dog, and mouse.
- Use machine 3 to convert the mouse into a cat and dog, so the result is 2 cats and 2 dogs
- Use Machine 1 in reverse to convert a dog to a cat, so the result is 3 cats and 1 dog

Discussion / questions: OK, so here the value of 3 cats and 1 dog is $4, so at least the invariant we looked at in Questions 2.1 and 2.2 could POSSIBLY allow us to convert the one cat into 3 cats and one dog. But you need to actually show HOW to do it in order to conclude that the answer is yes. Notice also that in one more step (using Machine 1 in reverse again), we could get to 4 cats.

Question 2.4

Can she convert it into one mouse? (If so, show how. If not, explain why not.)

Answer 2.4: NO. Use the same invariant as in Answer 2.1. 1 mouse would be worth $2, and $2 is not on the allowable list.

Question 2.5

Using these three Mad Vet machines, how many different "classes" of animals are there? Describe all of the classes.

Answer 2.5: There are THREE different "classes". Here's why. We know that a set of one cat is DIFFERENT than a set of two cats (the answer to Question 2.1 shows we can't get from one cat to two cats). Similarly, these are each different than a set of three cats. So "1 Cat", "2 Cats", and "3 Cats" are different classes of menageries (we can't use the machines to get from one to the other).

But now we claim that ANY collection of animals that the Mad Vet starts with can be changed to one of these three sets! Here's how. Take the starting collection. If there are any mice in the collection, use Machine 3 as many times as necessary to change all the mice into cats and dogs. So now the collection consists only of some number of cats and dogs. Now use Machine 1 to convert all the dogs to cats. So now the collection consists only of some number of cats.

In Question 2.3 we saw how to convert 1 cat to 4 cats. Working backwards, that means we can convert 4 cats to 1 cat as well. So now here's the idea. Take this collection of cats. If you have at least 4 cats, trade 4 cats in for 1 cat. If you still have at least four cats, do it again. Keep going. In the end you will have either 1 cat, or 2 cats, or 3 cats in your menagerie.

Remark: One could ask a question similar to Question 2.5 about Puzzle 1; that is, how many different "classes" of animals are there in Puzzle 1? It's not hard to show that there are at least 6 different classes (use the invariant described in Answer 1.2). But there are actually 30 different classes. This conclusion is probably beyond the scope of what most middle schoolers can show (although I have met one such middle schooler who figured it out!)