Comprehensive Exam - Linear Algebra
Spring 2009
Show All Your Work

NOTE: F denotes a field which is either the real or complex numbers.

1. Let P_6 denote the vector space over F of polynomials with degree at most 6. Let U denote the subset of P_6 consisting of polynomial p such that $p(1) = p(2) = p(3) = 0$. Show that U is a subspace of P_6. Find a basis for U, and its dimension.

2. Suppose V is a finite dimensional vector space over F, and $S_n : V \to V$ is an invertible linear map for $n = 1, \ldots, N$. Show that $T = S_1 \cdots S_N$ is invertible, and find T^{-1}.

3. Let A be the 5×5 matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

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(a) What is the range of A?

(b) Find the eigenvalues of A, and a basis of eigenvectors for R^5.

4. Suppose V is a vector space over F. Prove that the union $U_1 \cup U_2$ of two subspaces of V is a subspace if and only if one of the subspaces is contained in the other.

5. (a) Suppose V is an inner product space, and $v_1, v_2, \ldots$ is an orthonormal set with

$$\langle v_m, v_n \rangle = \left\{ \begin{matrix} 1, & m = n, \\ 0, & m \neq n \end{matrix} \right\}$$

Prove that the vectors $\{v_1, v_2, \ldots\}$ are linearly independent.

(b) For $n = 1, 2, 3, \ldots$, the functions $\sin(n\pi x)$ satisfy the identities

$$\int_0^1 \sin(m\pi x) \sin(n\pi x)\, dx = \left\{ \begin{matrix} 1/2, & m = n, \\ 0, & m \neq n \end{matrix} \right\}.$$

If

$$f(x) = \sum_{k=1}^{N} b_k \sin(k\pi x), \quad 0 \leq x \leq 1,$$

find a formula for b_k?

6. Suppose V is a finite dimensional vector space over F, and A and B are linear maps from V to V. Assume there is an invertible linear map $S : V \to V$ such that

$$B = S^{-1}AS.$$

(a) Prove that A and B have the same eigenvalues.

(b) Prove that if $p(x)$ is any polynomial with coefficients in F, then

$$p(B) = S^{-1}p(A)S.$$

7. Suppose V is an inner product space, and $T : V \to V$ is linear. If T^* is the adjoint of T, show that the null space of T^* is $range(T)^{\perp}$.

8. Suppose V is a finite dimensional vector space over F, and $T : V \to V$ is a nilpotent operator on V, that is, $T^k = 0$ for some positive integer k.

(a) What are the possible eigenvalues for T? Give a proof.

(b) For any positive integer N, give an example of a nilpotent operator T on F^N for which the span of the eigenvectors is a one-dimensional subspace.

9. Suppose R is a 3×3 real orthogonal (unitary) matrix, with $\det(R) = 1$. Show that 1 is an eigenvalue of R.

10. Suppose $x_0, \ldots, x_N$ are **distinct** elements of F. Consider the $(N+1) \times (N+1)$ matrix

$$A_N(x) = \begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^N \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & x_{N-1} & x_{N-1}^2 & \ldots & x_{N-1}^N \\ 1 & x & x^2 & \ldots & x^N \end{pmatrix}$$

with the variable x in the last row. Define

$$V_N(x) = \det(A_N(x)).$$

(a) Show that $V_N(x)$ is a polynomial of degree at most$N$, with roots $x_0, \ldots, x_{N-1}$.

(b) By induction, or otherwise, show that $V_N(x_N) \neq 0$.