

Comprehensive Exam – Linear Algebra
January 2008

1. Determine if each of the following statements is TRUE or FALSE by giving a short proof or a counterexample.
 - (a) Given any subspace W of a finite dimensional vector space V , there exists a subspace W' such that $W \oplus W' = V$.
 - (b) If T is linear operator on a vector space V and β is a linearly independent set of V , then $T(\beta)$ is linearly independent.
 - (c) A matrix A , which is similar to a self-adjoint matrix B , is also self-adjoint.

2. Let $P_3(\mathbb{R})$ be the space of real polynomials of degree at most 3 and $T : P_3(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ be given by $Tf = -f''$. Determine the null space $N(T)$, the range $R(T)$, and find the matrix representation $[T]_\beta$ in the standard basis β for $P_3(\mathbb{R})$.

3.
 - (i) State the Cayley-Hamilton theorem for matrices and then use it to prove that any $n \times n$ upper triangular matrix with all diagonal entries equal to zero is nilpotent.
[By definition, a square matrix A is nilpotent if $A^p = 0$ for some positive integer p .]
 - (ii) State Schur's theorem and use it to prove the converse of (i): that any nilpotent matrix A with complex entries is similar to an upper triangular matrix with all diagonal entries equal to zero.

4. Given an unitary operator U on an inner product space V , prove that
 - (a) all eigenvalues of U have absolute value 1.
 - (b) any two eigenvectors, corresponding to distinct eigenvalues of U , are orthogonal.

5. A linear operator T on a finite dimensional inner product space V is called an orthogonal projection if T is a projection operator and $N(T) = R(T)^\perp$.
 - (a) If T is an orthogonal projection, prove that $\|T(x)\| \leq \|x\|$ for all $x \in V$. Give an example of a projection for which this inequality does not hold (for some $x \in V$).
 - (b) Prove the converse of part (a): That is, suppose that T is a projection such that $\|T(x)\| \leq \|x\|$ for all $x \in V$, then prove that T must be an orthogonal projection.
[Hint: Take $x = z - \frac{\langle z, y \rangle}{\|y\|^2}y$, for $y \in N(T)$ and $z \in R(T)$.]

6. Suppose A and B are two $n \times n$ matrices with complex entries with the same characteristic polynomials and same minimal polynomials.
 - (a) If $n = 3$ show that A and B must be similar.
 - (b) Show that there are non-similar A and B with this property when $n > 3$.

[Recall that the largest integer d such that $(t - \lambda)^d$ divides the minimal polynomial of a matrix equals the largest size of a Jordan block corresponding to the eigenvalue λ appearing in the Jordan canonical form of that matrix.]