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cmssdc10 at 15pt

1. Assume that the sequence a_n converges to some a in $(0,1)$.

(a) Show that there exists a natural N such that for all $n \geq N$ we have a_n in $(0,1)$.
(b) Is the statement in (a) true if we replace $(0,1)$ by $(0,1]$. Prove it or provide a counter-example.

2. Let $g \geq 0$ be continuous on $[-1,1]$ and assume that $g(0) > 0$.
(a) Show that there exists $\delta > 0$ such for all x in $(-\delta, +\delta)$ $g(x) > 0$.
(b) Show that g is Riemann integrable on $[-1,1]$ and that

$$\int_{-1}^{1} g(x)dx > 0.$$

(c) Find a function g on $[-1,1]$ which is not continuous, $g \geq 0$ with $g(0) > 0$, and such that

$$\int_{-1}^{1} g(x)dx = 0.$$

3. For every natural number n, let

$$I_n = \int_0^1 \frac{x^n}{2-x} dx.$$

(a) Show that for every n

$$I_n \leq \frac{1}{n+1}.$$

(b) Show that I_n is convergent and find its limit. **4.**

(a) Show that for every x in $(0,1)$ there is a c in $(0,1)$ such that

$$\ln(1+x) = \frac{1}{1+c}x.$$

(b) Prove that

$$\ln(1+x) \leq x, \ x \in (0,1).$$

5. Let f be defined on the reals. Assume that $f(0) = 0$ and that f is differentiable at 0.
(a) Does $f(x)/x$ have a limit as x approaches 0?

(b) Does $f(x^2)/x$ have a limit as x approaches 0?

Part 2

6. a) Define what it means for a sequence $\{x_n\}$ from a metric space (S,d) to converge to $X \in S$. Define what it means for a sequence $\{x_n\}$ from a metric space (S,d) to be a Cauchy sequence.

b) Prove that every convergent sequence is a Cauchy sequence.

c) Give an example of a metric space which has a Cauchy seqeunce that does not converge.

7. State and prove the integral test for convergence of an infinite series

$$\sum_{n=1}^{\infty} a_n, \quad a_n \geq 0.$$

8. Determine whether the following series converge. Justify your answers.

$$a) \ \sum_{k=1}^{\infty} \frac{k^2+2}{k^4+8k^3}, \quad b) \ \sum_{k=1}^{\infty} \frac{\cos(1/k)}{k},$$

$$c) \ \sum_{k=1}^{\infty} \frac{k!}{k^k}, \quad d) \ \sum_{k=2}^{\infty} \frac{1}{\log(k)^2}.$$

9. Recall that

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}\, dt.$$

Use the geometric series to find a power series representation for $\frac{1}{1+t^2}$ in powers of t. Integrate the series term by term to get a power series for $\tan^{-1}(x)$. How do you justify this computation? What is the radius of convergence?

10. Suppose that A and B are disjoint compact subsets of R^N, and let $d(x,y)$ denote the standard Euclidean metric. If

$$M = \inf d(x,y), \quad x \in A, y \in B,$$

prove that $M > 0$. (Hint: Pick a sequence (x_n, y_n) such that $M = \lim_{n\to\infty} d(x_n,y_n)$.)