Comprehensive Exam – Linear Algebra
Spring 2007

1. Determine if each of the following statements is TRUE or FALSE by giving a short proof or a counterexample.

   (a) If A is a nonsingular square matrix which is diagonalizable, then so is A^{-1}.

   (b) If a linear operator T on a *real* inner product space V satisfies $< Tv, v >= 0$ for all $v \in V$, then T must be the zero transformation.

   (c) The matrices $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ are similar.

2. For a fixed $a \in \mathbb{R}$, consider the subspace $W = \{f \in P_n(\mathbb{R}) |\, f(a) = f'(a) = 0\}$ of $P_n(\mathbb{R})$, the space of real polynomials with degree at most n. Determine the dimension of W and write a basis for W.

3. Given a linear operator T on a finite dimensional vector space V, satisfying $T^2 = T$.

   (a) Using the dimension theorem, show that $N(T) \oplus R(T) = V$.

   (b) Identify all eigenvalues of T and the corresponding eigenspaces and show that T is diagonalizable.

4. (i) If A and Q are unitary matrices, show that $U = Q^{-1}AQ$ is also unitary.

   (ii) If a unitary matrix U is also upper triangular, show that U must be diagonal.

   (iii) State Schur's theorem and apply it together with parts (i) and (ii) to show directly that any unitary operator T on a finite dimensional *complex* inner product space is diagonalizable.

   (iv) Does the conclusion in part (iii) remain valid if V is a *real* inner product space? Justify your answer.

5. Given the 4×4 matrix $A = \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$,

   (a) Find the Jordan canonical form J of the matrix A [You do NOT need to compute generalized eigenvectors in this part].

   (b) Determine the minimal polynomial of A.

   (c) Show that A is nonsingular and express A^{-1} as a polynomial (of least degree) of A.

   (d) Find an invertible matrix Q such that $A = QJQ^{-1}$.