Comprehensive Exam – Linear Algebra
Spring 2006

1. (a) Suppose $S = \{x_1, x_2, \dots, x_m\}$ is a linearly independent subset of a vector space V and $Y = \{y_1, y_2, \dots, y_n\}$ spans V. Show that $m \leq n$.

(b) Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. Prove that $W_1 + W_2$ and $W_1 \cap W_2$ are subspaces of V and that $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

2. Let $v_1, v_2, \dots, v_{n-1} \in R^n$ and let $T : R^n \to R$ be a map defined by $T(x) = \det[v_1, v_2, \dots, v_{n-1}, x]$, the determinant of the $n \times n$ matrix whose columns are given by the vectors $v_1, v_2, \dots, v_{n-1}$ and x, respectively.

(i) Show that T is a linear transformation.

(ii) Prove that $T \neq T_0$ if and only if $S = \{v_1, v_2, \dots, v_{n-1}\}$ is a linearly independent set.

(iii) If $T \neq T_0$ – the null transformation, then prove that S is a basis of the null space $N(T)$.

3. (a) Let $A \in M_{n\times n}(C)$ be an upper triangular matrix whose diagonal entries are *all equal.* Show that A is diagonalizable if and only if A is a diagonal matrix.

(b) Consider the function $f(\mathbf{x}) = 2x_1x_2 + x_3^2$ where the column vector $\mathbf{x} = (x_1, x_2, x_3)^T \in R^3$.

(i) Find a 3×3 symmetric matrix A such that $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

(ii) Obtain an orthogonal matrix Q such that $A = QDQ^{-1}$ where $D = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, and λ_i's are the eigenvalues of A. Then show that $f(\mathbf{x}) = \mathbf{y}^T D\mathbf{y}$ where $\mathbf{x} = Q\mathbf{y}$.

(iii) Prove that on the unit sphere $\mathbf{x}^T\mathbf{x} = 1$, the maximum and minimum values of the function $f(\mathbf{x})$ occur along the eigenvectors corresponding to the largest and smallest eigenvalues respectively. Compute the maximum and minimum values of $f(\mathbf{x})$.

4. Let $T : V \to V$ be a self-adjoint linear operator on a finite-dimensional inner product space V.

(a) Prove that there exists an orthonormal basis for V consisting of eigenvectors of T.

(b) A self-adjoint linear operator T is called *positive definite* if for all non-zero vectors $v \in V$, the inner product $<Tv, v> > 0$. Show that T is positive definite if and only if all the eigenvalues of T are positive.

5. Consider the vector space V spanned by the basis set $\beta = \{e^x, xe^x, e^{-x}, xe^{-x}\}$ of real valued functions. Let $T : V \to V$ be a linear operator defined by $T(f) = \frac{df}{dx}$, $f \in V$.

(i) Calculate the characteristic polynomial of T. Hence show that that V is the solution space of a fourth order, linear, ordinary differential equation. Find this differential equation explicitly.

(ii) Find a Jordan canonical form J and a Jordan canonical basis γ for T.