Spring 2005

Master of Science Exam in Applied Mathematics - Analysis - January 28, 2005

There are 9 problems. All solutions submitted will be graded and the best 6 will be used for the grade.

1. Let K be a compact set in $\mathbb{R}^2$ and let f be a continuous function on K. Show that there is an $x \in K$ for which

$$f(x) = \sup\{f(t) : t \in K\}.$$

2. Let

$$f_n(x) = \frac{1}{x^2 + n^2}, \quad x \in \mathbb{R}.$$

Prove or disprove : f_n converges uniformly on $\mathbb{R}$.

3. Suppose that f_n is a sequence of continuous functions on $[0,1]$ which converges pointwise to a function f :

$$\lim_n f_n(x) = f(x), \text{ all } x \in [0,1].$$

a) If the convergence is uniform, prove that

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)dx.$$

b) Give an example for which

$$\int_0^1 f(x)dx \neq \lim_{n\to\infty} \int_0^1 f_n(x)dx.$$

4. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ and define

$$g(x,y,z) = z - h(x,y),$$
$$F(x,y,z) = f(x,y,h(x,y)).$$

a) Show that the problem of finding local minimums of f subject to the constraint

$$g(x,y,z) = 0$$

is the same as finding local minimums of F.

b) Suppose we have a local minimum at (x_0, y_0, z_0), so that by a)

$$\nabla F(x_0, y_0, z_0) = 0$$

holds. Show that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

for a scalar λ.

5. Determine convergence or non convergence of the infinite series'

$$\sum_{k=2}^{\infty} \frac{(-1)^k (k+1)}{k^2 - 1}, \quad \sum_{n=1}^{\infty} \frac{10^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{n!}{n^{10}}.$$