Comprehensive Exam – Linear Algebra
Spring 2004

1. R^3 is a real vector space whose elements are the ordered triplets of real numbers: $x = (x_1, x_2, x_3)$, $x \in R^3$. Consider the following subsets of R^3:

$W_1 = \{x \in R^3 : x = (0, x_2, x_3)\}$, $W_2 = \{x \in R^3 : x = (x_1, 0, x_3)\}$, $W_3 = \{x \in R^3 : x = (x_1, x_2, 0)\}$.

(a) Prove whether each of the following sets are *subspaces* of R^3:

$$(i)\ W_i,\ i = 1,2,3 \qquad (ii)\ W_i \cap W_j,\ i \neq j \qquad (iii)\ W_i \cup W_j,\ i \neq j \qquad (iv)\ W_1 \cap W_2 \cap W_3.$$

(b) Show that $R^3 = W_{12} \oplus W_{23} \oplus W_{31}$, where $W_{ij} = W_i \cap W_j$.

2. Consider the real vector space P(R) of all polynomials in x with real coefficients. Note that P(R) is not finite dimensional. For each $j \geq 1$, define the linear transformations D^j: P(R) $\to$ P(R) by $D^j(p(x)) := \dfrac{d^j p(x)}{dx^j}$, $p(x) \in$ P(R).

(a) Show that the linear transformation D^j is *onto.*
(b) Find a basis and the dimension of the null space $N(D^j)$.
(c) Prove that for any $n \geq 1$, $S = \{D^1, D^2, \ldots, D^n\}$ is a linearly *independent* subset of the vector space of all linear transformations from P(R) into P(R).

3. Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is an ordered, *orthonormal* basis for an inner product space V over a field F. For each $i = 1, 2, \ldots n$, define the functions T_i: V $\to$ V by

$$T_i(x) := \langle x, v_i \rangle\, v_i,\ x \in V,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on V.

(a) Show that T_i is a linear transformation. Determine the null space $N(T_i)$.
(b) Prove that the orthogonal complement of the range $R(T_i)^{\perp} = N(T_i)$.
(c) Let T_i^* denote the *adjoint* of T_i, and let T_0 and I denote the zero and identity transformations, respectively, on V. Then establish the following properties of T_i:

$$(i)\ T_i^* = T_i \qquad (ii)\ T_i^2 = T_i \qquad (iii)\ T_iT_j = T_0,\ i \neq j \qquad (iv)\ \sum_{i=1}^{n} T_i = I.$$

(d) Suppose for a certain linear transformation T: V $\to$ V, the basis β given above, is a basis of eigenvectors corresponding to eigenvalues λ_i, $i = 1, 2, \ldots n$. Then prove that T is a *normal* linear transformation. Also, show that $T = \sum_{i=1}^{n} \lambda_i T_i$.

4. Suppose the *minimal* polynomial of a 6×6 matrix A is given by $m(t) = (t-1)^2(t-2)$.

(a) List the eigenvalues of A together with all possible algebraic multiplicities.
(b) Is A diagonalizable? Justify your answer.
(c) Show that A is invertible and express A^{-1} as a polynomial in A.
(d) Find the minimal polynomial of A^{-1}.

Cool, there's another problem on the back!

5. Given the 4×4 matrix

$$\mathrm{M} = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}.$$

(a) Find a Jordan canonical basis β associated with the matrix M.

(b) Determine a Jordan canonical form J for M and a matrix P such that $\mathrm{P}^{-1}\mathrm{MP} = \mathrm{J}$. You MUST verify the relation $\mathrm{P}^{-1}\mathrm{MP} = \mathrm{J}$ (with or without computing P^{-1}).

(c) A matrix exponential function is defined as $\Phi(t) = \sum_{k=0}^{\infty} \frac{(\mathrm{J}t)^k}{k!}$, where J is a constant $n \times n$ matrix.

(i) Show that $\frac{d}{dt}\Phi(t) = \mathrm{J}\Phi(t)$.

(ii) Compute $\Phi(t)$ explicitly, with the J obtained above in part (b).

Solns

1 (a) Let S be *any* of the sets listed. Need to check subspace property:

If $u, v \in S$ and $c, d \in R$, then $cu + dv \in S$

(i) $W_1 = \{(0, x_2, x_3)\}$ is a subspace, similarly W_2, W_3

(ii) $W_1 \cap W_2 = \{(0,0,x_3)\}$ is a subspace, similarly $W_2 \cap W_3$ and $W_1 \cap W_3$

(iii) $W_1 \cup W_2 = \{(0, x_2, x_3) \text{ or } (y_1, 0, y_3)\}$ is *not* a subspace since
$u + v = (0, x_2, \frac{x_3}{2}) + (x_1, 0, \frac{x_3}{2}) = (x_1, x_2, x_3) \in R^3$, but $u+v \notin W_1 \cup W_2$. Similarly the others.

(iv) $W_1 \cap W_2 \cap W_3 = \{(0,0,0)\}$ — subspace

(b) Direct Sum: $V = V_1 \oplus V_2 \oplus V_3$ iff (i) $V_i \cap V_j = \{(0,0,0)\}$, $i \neq j$ and
(ii) $V \ni v = v_1 + v_2 + v_3$ such that $v_i \in V_i$. Verify these.

(i) $W_{12} \cap W_{23} = W_{23} \cap W_{31} = W_{12} \cap W_{31} = \{(0,0,0)\}$

(ii) $v = (x_1, x_2, x_3) = \underset{W_{23}}{\underset{\cap}{(x_1,0,0)}} + \underset{W_{31}}{\underset{\cap}{(0,x_2,0)}} + \underset{W_1 \cap W_2 = W_{12}}{\underset{\cap}{(0,0,x_3)}}$

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2 (a) Show that the range $R(P(R)) = P(R)$. For example, think of
$P(R) = \text{span}(\{1, x, x^2, \cdots, x^n, \cdots\})$.

$D^j\{1, x, x^2, \cdots\} = \{\underbrace{0, 0, \cdots, 0}_{j-1 \text{ terms}}, j!, (j+1)!\,x, \frac{(j+2)!}{2!}x^2, \cdots \frac{n!}{(n-j)!}x^{n-j}, \cdots\}$ (they don't need to be this precise!)

Then clearly, $\text{span}(D^j\{1, x, x^2, \cdots\}) = \text{span}(\{1, x, x^2, \cdots\}) = P(R)$.

(b) A basis for $N(D^j) = \beta_j = \{1, x, x^2, \cdots x^{j-1}\}$, $\dim(N(D^j)) = \underline{j}$

(c) Let $D = c_1 D^1 + c_2 D^2 + \cdots + c_n D^n$, for real scalars $c_1, c_2 \cdots c_n$. Need to show
$D = \underset{\sim}{O}$ (null transform) $\Rightarrow c_1 = c_2 = \cdots = c_n = 0$.

$D = \underset{\sim}{O} \Rightarrow D(p) = 0$, for any $p \in P(R)$. Choosing *successively*,
$p = x$, $p = x^2$, $\cdots$ $p = x^{n-1}$, we obtain $c_1 = c_2 = \cdots = c_{n-1} = 0$, so $c_n = 0$ also. Thus for $n > 1$, S is lin indep. When $n = 1$, $S = \{D^1\}$ which is a singleton, and by defn, is lin indep.

③ (a) Verify linearity properties (i) $T_i(x_1+x_2) = T_i(x_1) + T_i(x_2)$ and (ii) $T_i(cx) = c\,T_i(x)$ for $x_1, x_2, x \in V$ and any scalar $c \in F$

Null space: $T_i(x) = \underset{\sim}{0} \Rightarrow \langle x, v_i\rangle = 0 \Rightarrow N(T_i) = \text{span}(\{v_j,\ j=1,2\cdots n,\ j\neq i\})$

(b) $R(T_i) = \text{span}(\{v_i\})$, $R(T_i)^\perp = \{y \in V \mid \langle y, v_i\rangle = 0\}$.

Clearly: from (a), $N(T_i) \subseteq R(T_i)^\perp$ since $\langle v_j, v_i\rangle = 0,\ j\neq i$

Moreover, if $y \in R(T_i)^\perp$, then $T_i(y) = \langle y, v_i\rangle v_i = 0 \Rightarrow y \in N(T_i)$.

Thus, $R(T_i)^\perp \subseteq N(T_i)$, and so $R(T_i)^\perp = N(T_i)$.

(c) (i) $\forall\ x, y \in V$ $\langle x, T_i^*(y)\rangle = \langle T_i(x), y\rangle = \langle x, v_i\rangle\langle v_i, y\rangle = \langle x, \langle y, v_i\rangle v_i\rangle$ (using symmetry: $\langle u, cv\rangle = \bar{c}\langle u, v\rangle$ and $\langle u, v\rangle = \overline{\langle v, u\rangle}$)

$\therefore \langle x, T_i^*(y)\rangle = \langle x, T_i(y)\rangle,\ \forall\ x, y \in V$

$\Rightarrow \underline{T_i = T_i^*}$

(ii) For any $x \in V$, $T_i^2(x) = T_i(T_i(x)) = T_i(\langle x, v_i\rangle v_i) = \langle x, v_i\rangle T_i(v_i)$, where $T_i(v_i) = v_i$

$\therefore T_i^2(x) = \langle x, v_i\rangle v_i = T_i(x) \Rightarrow \underline{T_i^2 = T_i}$

(iii) same as (ii), except $T_i(v_j) = 0,\ i\neq j$

(iv) Fourier expansion: In an orthonormal basis, $x = \sum_{i=1}^{n} c_i v_i$ where the scalars $c_i = \langle x, v_i\rangle$ (they need to show this).

$\therefore Ix = x = \sum_{i=1}^{n}\langle x, v_i\rangle v_i = \sum_{i=1}^{n} T_i(x)$, $\forall\ x \in V \Rightarrow \underline{I = \sum_{i=1}^{n} T_i}$

(d) Normal $\overset{\text{defn}}{\Longrightarrow}$ $T^*T = TT^*$, $T^* \to$ adjoint of T.

Using the orthonormal basis $\beta = \{v_1, v_2 \cdots v_n\}$, we can calculate

$\langle T^*(v_j), v_i\rangle = \langle v_j, T(v_i)\rangle = \langle v_j, \lambda_i v_i\rangle = \bar{\lambda}_i\langle v_j, v_i\rangle = \bar{\lambda}_i\delta_{ij}$

$\therefore T^*(v_j) = \bar{\lambda}_j v_j$. So T, T^* both, have diagonal matrix representatio[n] in the β-basis — $[T]_\beta = \text{diag}(\lambda_1, \lambda_2 \cdots \lambda_n)$ & $[T^*]_\beta = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \cdots \bar{\lambda}_n)$.

Hence $[T]_\beta[T^*]_\beta = [T^*]_\beta[T]_\beta \Rightarrow TT^* = T^*T$. (diag matrices commute)

(Alt. way): $\forall\ v_i \in \beta$, $TT^*(v_i) = \bar{\lambda}_i T(v_i) = \bar{\lambda}_i\lambda_i v_i$; $T^*T(v_i) = \lambda_i T^*(v_i) = \lambda_i\bar{\lambda}_i v_i$ $\Rightarrow \underline{TT^* = T^*T}$ since their action on each basis element are the same

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Finally, for any $x = \sum_{i=1}^{n} x_i v_i \in V$, $T(x) = \sum_{i=1}^{n} x_i T(v_i) = \sum_{i=1}^{n}\lambda_i x_i v_i$

But $x_i = \langle x, v_i\rangle$ (in an o.n. basis) $\therefore T(x) = \sum_{i=1}^{n}\lambda_i\langle x, v_i\rangle v_i = \sum_{i=1}^{n}\lambda_i T_i(x)$ $\Rightarrow T = \sum_{i=1}^{n}\lambda_i T_i$

④ (a) $(\lambda_1, \lambda_2) = (1, 2)$, $(m_1, m_2) = (2,4)$ or $(3,3)$ or $(4,2)$ or $(5,1)$ — possible algebraic mult.

(b) A is diagonalizable $\Longleftrightarrow$ $m(t) = (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_k)$ (Thm 7.16 p 520 of text) — simple linear factors

Since $m(t) = (t-1)^2(t-2)$, A is not diagonalizable.

(c) Eigvals of A are $\lambda = 1, 2 \Rightarrow \lambda = 0$ is not an eigvalue $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow A$ invertible

$m(A) = (A-I)^2(A-2I) = A^3 - 4A^2 + 5A - 2 = 0$ (multiply by A^{-1})

or, $A^2 - 4A + 5I = 2A^{-1} \Rightarrow A^{-1} = \frac{1}{2}A^2 - 2A + \frac{5}{2}I$

(d) $m(A) = A^3\underbrace{(I - 4A^{-1} + 5A^{-2} - 2A^{-3})}_{\text{call } h(A^{-1})} = 0 \Rightarrow h(A^{-1}) = 0.$

Note that $h(A^{-1})$ is the polynomial in A^{-1}, of least degree (=3), that vanishes. (For, if $\deg(h(A^{-1})) < 3$, then $\deg(m(A)) < 3$ also — contradiction, $m(A)$ is minimal.)

The minimal polynomial of A^{-1} is the monic polynomial of least degree that vanishes. Call this $n(A^{-1})$.

$$\therefore\ n(A^{-1}) = \frac{h(A^{-1})}{-2} = A^{-3} - \frac{5}{2}A^{-2} + 2A^{-1} - \frac{1}{2}I$$

⑤. parts (a, b) — refer to Example 2 (p 500) of text. Students may have different choices for β and P (book calls it Q), hence J.

Since the cols of P are eigvectors/gen eigvectors, they can verify $MP = PJ$ (by letting M act on each col of P) and do not need to compute P^{-1}. But if they don't think (remember) of it then they must compute P^{-1}.

(c) (i) $$\frac{d\Phi}{dt} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{J^k t^k}{k!}\right) = J\left(\sum_{k=1}^{\infty} \frac{J^{k-1}t^{k-1}}{(k-1)!}\right) = J\sum_{l=0}^{\infty}\frac{J^l t^l}{l!} = J\Phi(t)$$

(let $k-1 = l$)

(ii) $$J = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \operatorname{diag}(J_1, J_2, J_3). \quad \therefore\ \Phi(t) = \operatorname{diag}\left(\sum_{k=0} \frac{J_1^k t^k}{k!}, \underbrace{\sum \frac{J_2^k t^k}{k!}}_{e^{2t}}, \underbrace{\sum\frac{J_3^k t^k}{k!}}_{e^{3t}}\right)$$

(from part (b))

Note: $J_1 = 2I + N$ $(N^2 = N^3 = \cdots = 0)$, $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

$$\frac{J_1^k t^k}{k!} = \frac{t^k}{k!}(2I+N)^k = \frac{(2t)^k}{k!}I + \frac{(2t)^{k-1}}{(k-1)!}tN, \quad k \geq 1$$

$$\therefore\ \Phi(t) = \operatorname{diag}\left(e^{2t}(I+Nt), e^{2t}, e^{3t}\right) = \begin{pmatrix} e^{2t} & te^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{3t} \end{pmatrix}$$