# Comprehensive Exam – Linear Algebra  Spring 2001

1. In $\Re^2$, **N** is the usual natural basis $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$; let **B** be the basis $\left\{\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}$. **T** is a linear transformation on $\Re^2$ defined by its action on **N**: $\mathbf{T}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$ and $\mathbf{T}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$. $\mathbf{T}^2=\mathbf{T}\circ\mathbf{T}$.

   a) Find the matrix (often denoted ${}_{\mathbf{N}}[\mathbf{T}^2]_{\mathbf{N}}$) that represents $\mathbf{T}^2$ with respect to the basis **N**.
   b) Find the matrix (often denoted ${}_{\mathbf{B}}[\mathbf{T}^2]_{\mathbf{B}}$) that represents $\mathbf{T}^2$ with respect to the basis **B**.
   c) Find a basis **C** and a diagonal matrix D such that the matrix that represents **T** with respect to **C** (i.e., ${}_{\mathbf{C}}[\mathbf{T}]_{\mathbf{C}}$) is D. (Note that this question refers to **T**, not $\mathbf{T}^2$.)

2. Let $\mathcal{V}$ be a vector space and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly independent set of vectors in $\mathcal{V}$. If **T** is a non-singular linear transformation on $\mathcal{V}$, show that $\{\mathbf{Tv}_1, \mathbf{Tv}_2, \ldots, \mathbf{Tv}_k\}$ is also a linearly independent set. (Clearly indicate the step in your proof where you use the hypothesis that **T** is non–singular.)

3. Let $\mathcal{P}_n$ be the vector space over $\Re$ of all polynomials of degree $n$ or less ($n \geq 3$) with real coefficients. Let $f$ and $g \in \mathcal{P}_n$ be defined as $f(x) = x + 1$ and $g(x) = x - 1$.
   a) Prove: $f$ and $g$ are linearly independent in $\mathcal{P}_n$.
   b) Let $\mathcal{S}$ be the subspace of $\mathcal{P}_n$ spanned by $f$ and $g$. Define a linear transformation **T** on $\mathcal{P}_n$ such that the kernel (or null space) of **T** is $\mathcal{S}$.

4. Let $A = \begin{pmatrix}1 & -2 & 3 & 0\\ -2 & 4 & -1 & 1\\ 3 & -6 & 4 & 1\end{pmatrix}$.

   a) Find a basis for the row space of A.
   b) Find a basis for the column space of A.
   c) Find a basis for the null space of A.
   d) What is the *dimension* of the null space of $A^T$?

5. $A = \begin{pmatrix}6 & 4 & 10 & \_\\ 8 & 0 & 5 & 0\\ 0 & 0 & 3 & 0\\ 9 & 6 & 7 & \_\end{pmatrix}$, but note that two entries of the last column are missing. The characteristic polynomial of A is known to be $p_A(t) = t^4 - 12t^3 - 5t^2 + 96t$. What are the missing entries?

6. Let $\mathbf{C}^n$ be the vector space of $n$-tuples over **C**, the field of complex numbers, and let $\langle\bullet,\bullet\rangle$ be the usual inner product on $\mathbf{C}^n$. Let $\|\bullet\|$ be the vector norm induced by this inner product. I.e., $\|\mathbf{v}\| = \langle\mathbf{v},\mathbf{v}\rangle^{1/2}$. Let U be an $n \times n$ unitary matrix. Prove that for every $\mathbf{v} \in \mathbf{C}^n$, $\|U\mathbf{v}\| = \|\mathbf{v}\|$.

7. Let A, an $n \times n$ matrix with entries from **C**, the field of complex numbers, have the property that $A^3 = I_n$, where $I_n$ is the $n \times n$ identity matrix.
   a) What are the possible eigenvalues of A?
   b) Prove: A is diagonalizable.
   c) Suppose further that $n = 3$ and all entries of A are <u>real</u> numbers. List all possible non-similar Jordan Canonical Forms for A.