Comprehensive Exam in Analysis, Spring 2000

Do 3 of the following 4 problems on this page

1. Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$ be a uniformly continuous function from X to Y.

(a) If $\{x_n\}$ is a Cauchy sequence in X, then prove that $\{f(x_n)\}$ is a Cauchy sequence in Y.

(b) Therefore, by a suitable choice of a Cauchy sequence $\{x_n\}$ in $(0, 1)$, conclude that $f(x) = 1/x$ is not uniformly continuous from $(0, 1)$ to $\mathbf{R}$

2. Let (X, d) be a metric space.

(a) Give a definition of a compact subset K of X.

(b) Let K be a compact subset of X and let L be a closed subset of K. Prove that L is compact in X as well.

3. Let $a_n(x) = \frac{x^n}{1+n^2x}$. Prove that $\sum_{n=1}^{\infty} a_n(x)$ converges

(a) pointwise, and (b) uniformly on $[0, 1]$.

4. Let $f_n(x) = n^2x^n(1 - x)$, $0 \leq x \leq 1$.

(a) Compute $\lim_{n\to\infty} \int_0^1 f_n(x)dx$

(b) Compute the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$

(c) Does f_n converge uniformly to f on $[0, 1]$? Why or why not?