Master of Science Exam in Applied Mathematics
Analysis – August 19, 2005

There are 10 problems here. The best 7 will be used for the grade.

1. Consider the space of functions

$$\mathcal{C} = \mathcal{C}([0,1],\mathbb{R}) = \{f : f \text{ maps } [0,1] \to \mathbb{R} \text{ and } f \text{ is continuous}\}$$

and define

$$d(f,g) := \sup\{|f(x) - g(x)| : x \in [0,1]\}.$$

(a) Show that $d(f,g)$ is a metric on $\mathcal{C}$.

(b) Let

$$\mathcal{F} := \{f \in \mathcal{C} : 0 \le f(x) \le 1 \text{ for } x \in [0,1]\}.$$

Show that $\mathcal{F}$ is (i) bounded and (ii) closed as a set in the metric space $\mathcal{C}$ under the metric $d(f,g)$.

(c) Define a sequence of functions $\{f_n\}$ in $\mathcal{C}$ by

$$f_n(x) = x^n,\ x \in [0,1].$$

Show that there is *no* subsequence $\{f_{n_k}\}$ of the given sequence that converges in $(\mathcal{C}, d)$.

SOLUTION

(a) There are 4 properties to verify

1) $d(f,f) = 0$; this is clear.

2) $d(f,g) = 0 \Rightarrow f(x) = g(x)$, all x.

Proof : If not, there is a t for which $f(t) \ne g(t)$. Then

$$d(f,g) \ge |f(t) - g(t)| > 0$$

3) $d(f,g) = f(g,f)$; this is clear because $|f(x) - g(x)| = |g(x) - f(x)|$, all x.

4) The triangle inequality :

$$d(f,g) \le d(f,h) + d(h,g).$$

Proof: For all x,

$$|f(x) - g(x)| = |(f(x) - h(x)) + (h(x) - g(x))| \le |f(x) - h(x)| + |h(x) - g(x)| \le d(f,h) + d(h,g).$$

Hence

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\} \le d(f,h) + d(h,g).$$

b) (i) $\mathcal{F}$ is bounded because

$$d(f,g) \le d(f,0) + d(0,g) \le 2.$$

(ii) $\mathcal{F}$ is closed because if $(f_n) \subset \mathcal{F}$ ~~and~~

$$\lim_{n\to\infty} d(f_n,g) = 0 \Rightarrow \lim(f_n(x) - g(x)) = 0, \text{ all } x \Rightarrow \lim f_n(x) = g(x) \Rightarrow 0 \le g(x) \le 1.$$

c) If there is a subsequence f_{n_k} in $\mathcal{F}$ and a function $g \in \mathcal{F}$ for which

$$\lim_{k\to\infty} d(f_{n_k}, g) = 0$$

then for all x

$$\lim_{k\to\infty} |x^{n_k} - g(x)| = 0; \quad \lim_{k\to\infty} x^{n_k} = g(x).$$

However

$$\lim_{k\to\infty} x^{n_k} = \begin{cases} 0 & if \quad x \in [0,1) \\ 1 & if \quad x = 1 \end{cases}$$

and this function is not continuous on $[0,1]$.

2. Define a sequence $\{a_n\}$ in $[-1,1] \subset \mathbb{R}$ by $a_n = \sin(n)$. Even though there seems to be no apparent pattern in the values of this sequence, it must have a convergent subsequence. State the relevant theory that proves the existence of such a convergent subsequence.

SOLUTION The interval $[-1,1]$ is a compact subset of $\mathbb{R}$ and one property of compactness of a set in $\mathbb{R}$ is that every sequence in the set has a convergent subsequence.

3. Define $g_n : [0,1] \to \mathbb{R}$ by $g_n(x) = e^{-ne^x}$.

(a) Show that $\lim_{n\to\infty} g_n(x) = 0$ uniformly on $[0,1]$.

(b) Prove in addition that $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on $[0,1]$.

SOLUTION. (a) $e^{-ne^x} \le e^{-n}$, all $x \in [0,1]$ and $\lim_{n\to\infty} e^{-n} = 0$.

(b) $|g_n(x)| \le e^{-n}$ for all $x \in [0,1]$ and the geometric series $\sum_{n=1}^{\infty} e^{-n}$ converges, and so by W-M test $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly.

4. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x,y) = \sin(x/3) + \cos(y/3)$.

(a) Show that the gradient vector of partial derivatives $\nabla g = (\partial g/\partial x, \partial g/\partial y)$ satisfies $\|\nabla g\| \leq 1/2$ for all $(x,y) \in \mathbb{R}^2$, (the vector norm is the Euclidean norm).

(b) The Mean Value Theorem asserts that if a function $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives on all of $\mathbb{R}^2$ then for all $(x_0,y_0),(x,y) \in \mathbb{R}^2$ there exists $\theta = \theta(x_0,y_0,x,y) \in (0,1)$ such that

$$f(x,y) = f(x_0,y_0) + (\nabla f) \cdot (x - x_0, y - y_0),$$

where the dot product is indicated in the formula and where each partial derivative in the gradient vector $\nabla f = (\partial f/\partial x, \partial f/\partial y)$ is evaluated at the point $(x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)) \in \mathbb{R}^2$. Conclude that

$$|g(x,y) - g(x_0,y_0)| \leq (1/2)\|(x - x_0, y - y_0)\|$$

for all $(x_0,y_0),(x,y) \in \mathbb{R}^2$.

(c) Define also $h : \mathbb{R}^2 \to \mathbb{R}$ by $h(x,y) = \sin(x/5) + \cos(y/5)$, and define the mapping $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x,y) = (g(x,y), h(x,y))$. Let $(a_0,b_0) = (0,0) \in \mathbb{R}^2$ and inductively define $(a_{n+1}, b_{n+1}) = \mathbf{F}(a_n, b_n) \in \mathbb{R}^2$. Verify that the mapping $\mathbf{F}$ on $\mathbb{R}^2$ is indeed a contraction and so conclude by the Contraction Mapping Theorem (check the hypotheses please) that the sequence $\{(a_n, b_n)\}$ has a limit in $\mathbb{R}^2$.

5. Let $f : (0,1] \to \mathbb{R}$

(a) Define uniform continuity for f on $(0,1]$

(b) Assume f is uniformly continuous. Let $\{x_n\}$ be a Cauchy sequence in $(0,1]$. Show that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

SOLUTION (a) For any $\varepsilon > 0$ there is a $\delta > 0$ for which

$$x \in (0,1] \text{ and } t \in (0,1] \text{ and } |x - t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon.$$

(b) There is an N for which

$$n \geq m \geq N \Rightarrow |x_n - x_m| < \delta.$$

Hence

$$|f(x_n) - f(x_m)| < \varepsilon.$$

6. Define f on $\mathbb{R}$ by

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(a) Show that f is continuous on $\mathbb{R}$.

(b) Show that $f'(0)$ exists and find $f'(0)$.

Hint : One approach is l'Hospital's rule.

SOLUTION. (a). Using l'Hospital's rule,

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = \cos(0) = 1.$$

This shows that f is continuous at 0; for $x \neq 0$, continuity follows from the continuity of $\sin(x)$ and of x.

(b) The difference quotient for $f'(0)$ is

$$\frac{\frac{\sin x}{x} - 1}{x} = \frac{\sin(x) - x}{x^2}$$

and using l'Hospital's rule twice :

$$\lim_{x\to 0} \frac{\frac{\sin x}{x} - 1}{x} = \lim_{x\to 0} \frac{\sin(x) - x}{x^2} = \lim_{x\to 0} \frac{\cos(x) - 1}{2x} = \lim_{x\to 0} \frac{-\sin(x)}{2} = 0.$$

Thus $f'(0)$ exists and

$$f'(0) = 0.$$

The $x \neq 0$ $f(x)$ is the quotient of functions having a derivative and so has a derivative.

For

7. Let K be a compact set in a metric space (X,d) and let f be a continuous real valued functon on (X,d).

(a) Prove that there is an $x \in K$ for which

$$f(x) = \sup\{f(t) : t \in K\}.$$

(b) Give an example of a set $K \subset \mathbb{R}$ and a function f on K for which the assertion fails.

<u>SOLUTION</u> (a) There is a sequence t_k in K for which

$$\lim_{k\to 0} f(t_k) = \sup\{f(t) : t \in K\}.$$

Since K is compact there is a subsequence t_{k_n} which converges, say

$$x = \lim_{n\to\infty} t_{k_n}.$$

By continuity of f we have

$$f(x) = \lim_{n\to 0} f(t_{k_n}) = \lim_{k\to 0} f(t_k) = \sup\{f(t) : t \in K\}.$$

(b) Let $K = \mathbb{R}$. Let

$$f(x) = 1 - \frac{1}{|x|+1}.$$

Then $f(x) < 1$ for all $x \in \mathbb{R}$ but

$$\sup\{f(t) : t \in K\} = 1.$$

8. One form of completeness of the real numbers $\mathbb{R}$ is that every bounded increasing sequence converges. Use this property to prove that every Cauchy sequence in $\mathbb{R}$ converges.

<u>SOLUTION.</u> Let (x_n) be a Cauchy sequence. First, there is a positive integer N for which

$$n \geq N \Rightarrow |x_n - x_N| < 1 \Rightarrow |x_n| < 1 + |x_N|.$$

Hence the sequence (x_n) is bounded. Define

$$y_k = \sup\{x_n : n \leq k\}.$$

The sequence (y_k) is increasing and so $z = \lim_{k\to\infty} y_k$ exists. For $\varepsilon > 0$, there is a K for which

$$k \geq K \Rightarrow |y_k - z| < \frac{\varepsilon}{2}.$$

Since (x_n) is a Cauchy sequence, there is an $N \geq K$ for which

$$n,k \geq N \Rightarrow |x_n - y_k| < \frac{\varepsilon}{2}.$$

Hence

$$n \geq N \Rightarrow |x_n - z| \leq |x_n - y_k| + |y_k - z| < \varepsilon$$

showing that

$$\lim_{n\to\infty} x_n = z.$$

9. Find the radius of convergence of each power series $\sum_{n=0}^{\infty} a_n x^n$.

$$\left[\quad a)\ a_n = n \quad b)\ a_n = \begin{cases} 0 & if\ \ n = 0 \\ \frac{1}{n} & if\ \ n > 0 \end{cases} \quad c)\ a_n = \begin{cases} 1 & if\ \ n = 2^k, \text{ some } k \geq 0 \\ 0 & if\ \ \ \ \ \ otherwise \end{cases} \quad \right]$$

SOLUTION (a) $r = 1$ by ratio test.

(b) $r = 1$ by ration test.

(c) $r = 1$ by root test :

$$r = \frac{1}{\limsup_{n\to\infty} (a_n)^{\frac{1}{n}}} = 1.$$

10. Let f be a function with domain $D \subset \mathbb{R}^2$, range $\mathbb{R}^2$, and defined by

$$f(x,y) = \left(\frac{x}{y}, \frac{y}{x}\right).$$

a) What is the natural domain D of f?

b) The local inverse mapping theorem applies to f. Find the set J for which the theorem guarantees a local inverse.

SOLUTION. (a) $D = \{(x,y) : x \neq 0 \text{ and } y \neq 0\}$.

(b) Let

$$f(x,y) = (u(x,y), v(x,y))$$

The derivatives

$$\frac{\partial u}{\partial x} = \frac{1}{y}, \ \frac{\partial u}{\partial y} = -\frac{x}{y^2}$$

$$\frac{\partial v}{\partial x} = -\frac{y}{x^2}, \ \frac{\partial u}{\partial y} = \frac{1}{x}$$

for $(x,y) \in D$ the Jacobian matrix is thus

$$M(f)(x,y) = \begin{bmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}$$

and the determinant is :

$$\det[M(f)(x,y)] = \det\begin{bmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix} = \frac{1}{xy} - \frac{x}{y^2}\frac{y}{x^2} = 0.$$

Thus the local inverse mapping does not apply for every $(x,y) \in D$. So the set is

$$J = \emptyset.$$