Fall 2004

Master of Science Exam in Applied Mathematics - Analysis - August 20, 2004.

1. There are various ways to construct the real numbers R from the rational numbers Q. In any of these methods the two properties

C : Every Cauchy Sequence in R Converges,

L : Every Bounded Set in R has a lub (Least Upper Bound)

are important. Show that $L \Rightarrow C$. You may, of course, use the field and order axioms for R.

2. One way to define compactness of a set in R is :

A set $K \subset R$ is compact if every sequence in K has a subsequence which converges to a point of K.

Use this definition and let f be a continuous function on K. Show that there is a point $x \in K$ for which

$$f(x) = lub\{f(t) : t \in K\}.$$

3. Suppose that $(a_k)_{k=1}^{\infty}$ is a sequence of non-negative real numbers and that $(b_k)_{k=1}^{\infty}$ is a sequence of real numbers for which

$$|b_k| \leq a_k.$$

If the infinite series $\sum_{k=1}^{\infty} a_k$ converges, show that $\sum_{k=1}^{\infty} b_k$ converges.

4. In the linear space

$$C[-1,1] = \{Continuous\ functions \quad f : [-1,1] \to R\}$$

define

$$\|f\| = \max\{|f(x)| : x \in [-1,1]\}.$$

a) Prove that

$$\rho(f,g) = \|f-g\|$$

defines a metric on $C[-1,1]$.

b) Using the fact that the metric space $(C[-1,1],\rho)$ is complete, show that the set of power series'

$$P = \{f : f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ has radius of convergence } 2\}$$

is a closed subset of $(C[-1,1],\rho)$.